Interim and Long-Run Dynamics in the Evolution of Conventions

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Introduction

theory of the evolution of conventions: Markov process with strong forces such as learning and weak forces such as mutations

analyze limit: equilibria of the game appear as irreducible classes of the Markov process

- near the limit process is ergodic and puts positive weight on all states.
- most weight on particular irreducible classes of the limit
- characterize which ones: method of least cost trees

here: analyzes the dynamics (transitions))

use the dynamics to give a simple characterization of ergodic distribution

illustrate the method with an application to the fall of hegemonies

Illustrative Example

2x2 symmetric coordination game with actions G, B

	G	B
G	2,2	0,0
B	0.0	1,1

two pure Nash equilibria at $GG$and BB and mixed at $1/3\ G$

Evolutionary Context

five players, state of the system is number of players playing G

state space Z has $N = 6$ states

each period one player chosen at random to make a move

behavior rule or deterministic dynamic represents rational learning: choose a best response to the actions of the opposing players against whom you will be randomly matched

independent trembles or mutations:

probability $1 - \epsilon$ behavior rule followed

with probability ϵ choice is uniform and random over all actions

Transition Matrix

$$P_\epsilon = \begin{pmatrix} 1-\frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 & 0 & 0 \\ \left(\frac{1}{5}-\frac{\epsilon}{10}\right) & \left(\frac{4}{5}-\frac{3\epsilon}{10}\right) & \frac{4\epsilon}{10} & 0 & 0 & 0 \\ 0 & \left(\frac{2}{5}-\frac{2\epsilon}{10}\right) & \frac{\epsilon}{2} & \left(\frac{3}{5}-\frac{3\epsilon}{10}\right) & 0 & 0 \\ 0 & 0 & \frac{3\epsilon}{10} & \left(\frac{3}{5}-\frac{\epsilon}{10}\right) & \left(\frac{2}{5}-\frac{2\epsilon}{10}\right) & 0 \\ 0 & 0 & 0 & \frac{4\epsilon}{10} & \left(\frac{4}{5}-\frac{3\epsilon}{10}\right) & \left(\frac{1}{5}-\frac{\epsilon}{10}\right) \\ 0 & 0 & 0 & 0 & \frac{\epsilon}{2} & 1-\frac{\epsilon}{2} \end{pmatrix}$$

$\epsilon = 0$ two irreducible classes $\Omega = \{\{0\}, \{5\}\}$ corresponding to the pure Nash equilibria of the game

basin of points for which probability of reaching $\{0\}$ is one is $\{0\}, \{1\}$; *basin* of $\{5\}$ is $\{3\}, \{4\}, \{5\}$

$\{2\}$ is in *outer basin* of both $\{0\}$ and $\{5\}$ both reached with positive probability from that point

Positive ϵ

typically or *most of the time* meaning in the limit as $\epsilon \to 0$

From Young or Ellison the system will spend most of the time at $\{5\}$

because of a special property (radius greater than co-radius) waiting times also known from Ellison: from $\{0\}$to $\{5\}$ roughly ϵ^{-2}

from $\{5\}$ to $\{0\}$ roughly ϵ^{-3}

another special property: birth-death process ergodic distribution can be explicitly computed

New Results

- When the transition from $\{0\}$ to $\{5\}$ takes place typically once the state $\{2\}$ is reached, there is no return to the state $\{1\}$ and the transition is very fast.
- When the transition from $\{5\}$ to $\{0\}$ takes place typically the states $\{4\}, \{3\}$ are reached in that order and once the state $\{3\}$ is reached there is no return to the state $\{4\}$ and once $\{2\}$ is reached there is no return to the state $\{3\}$ and the transition is very fast.
- Starting at $\{0\}$, before $\{5\}$ is reached the system will spend most of the time at $\{0\}$ but will many times reach the state $\{1\}$ for brief periods
- Starting at $\{5\}$, before the state $\{0\}$ is reached the system will spend most of the time at $\{5\}$ but will many times reach the states $\{4\}, \{3\}$ for brief periods
- The state $\{4\}$ will occur roughly as often as the state $\{0\}$ but while \$ $\{0\}$ will be seen for long stretches of time, the state $\{4\}$ will be seen frequently but only briefly before reverting to $\{5\}$

The Model

a finite state space Z with N elements

a family P_ϵ of Markov chains on Z indexed by $0 \leq \epsilon < 1$

two regularity conditions:

- $\lim_{\epsilon \to 0} P_\epsilon = P_0$
- there exists a resistance function $0 \leq r(x, z) \leq \infty$ and constants $0 < C \leq 1 \leq D < \infty$ such that $C\epsilon^{r(x,z)} \leq P_\epsilon(z|x) \leq D\epsilon^{r(x,z)}$

Resistances in the Example

$$r = \begin{pmatrix} 0 & 1 & \infty & \infty & \infty & \infty \\ 0 & 0 & 1 & \infty & \infty & \infty \\ \infty & 0 & 1 & 0 & \infty & \infty \\ \infty & \infty & 1 & 0 & 0 & \infty \\ \infty & \infty & \infty & 1 & 0 & 0 \\ \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

Irreducible Classes, Paths and Transitions

Ω union of the irreducible classes of P_0

$\Omega(x)$ for the irreducible class containing x where $\Omega(x) = \emptyset$ if x is not part of an irreducible class

path a a finite sequence $(z_0, z_1, \ldots, z_t)$ of points in Z

$t(a)$ number of transitions

resistance of the path $r(z_0, z_1, \ldots, z_t) \equiv r(z_0, z_1) + r(z_1, z_2) + \ldots + r(z_{t-1}, z_t)$ with the convention that for the *trivial path* with $t(a) = 0$ then $r(a) = 0$

Well Known Properties

Non-empty irreducible classes $\Omega(x) \neq \emptyset$ characterized by property: from any point $y \in \Omega(x)$ positive probability path to any other point $z \in \Omega(x)$ and every positive probability path starting at y must lie entirely within $\Omega(x)$. May equally say zero resistance instead of positive probability.

Comprehensive Sets

a set W is *comprehensive*

for any point $z \in Z$ there is a positive probability (zero resistance) path to some point in W

so Ω is comprehensive; more generally

Theorem: *A set W is comprehensive if and only if it contains at least one point from every non-empty irreducible class.*

Concept of Direct Routes

a *forbidden set* W for a path a is a set that the path does not touch except possibly at the beginning and end

given an initial point $x \in Z$ and sets $W \subseteq Z$ and $B \subseteq W$, we call a non-trivial path from x to B with forbidden set W a *direct route* if W is comprehensive and the path has positive probability for $\epsilon > 0$

For each x, B and comprehensive $W \supseteq B$ there is a set $\mathcal{A}(x, B, W)$ of direct routes from x to B with forbidden set W.

Motivation

how do we go (not necessarily directly) from one non-empty irreducible class $\Omega(x)$to a different non-empty irreducible class $\Omega(y)$?

impossible when $\epsilon = 0$

may be possible when $\epsilon > 0$ however:

to leave $\Omega(x)$ to get to $\Omega(y)$ at some point the path must leave $\Omega(x)$ and then hit some point in Ω, say a point in $\Omega(z)$

in other words a direct route from some $x' \in \Omega(x)$ to some set $B = \Omega(z)$ with forbidden set $W = \Omega$

hence while direct routes are improbable they are important because they are needed to move from one irreducible class to another

Intuition for Properties of Direct Routes

if a point in an irreducible class is hit then it is very likely that the path will then linger in that irreducible class passing through every point in the class many times

hence there should be a sense in which paths that do not hit a comprehensive set are quick: they cannot linger in an irreducible class for if they did so they would have to hit every point in the class many times, thus touching W

direct routes like the hare in the story of the tortoise and the hare.

They must get to the destination quickly – if they do not they will fall into the forbidden set

Questions about Direct Routes

how likely is the set of direct routes $\mathcal{A}(x, B, W)$

which paths in $\mathcal{A}(x, B, W)$ are most likely, what are these paths like and how long are they?

Results on Direct Routes

we have not assumed P_ϵ is ergodic - so to avoid triviality, we assume that $\mathcal{A}(x, B, W) \neq \emptyset$

important fact:

$r(A) = \min_{a \in A} r(a)$ is well-defined (and finite)

also define $t(A) = \min\{t(a) | a \in A, r(a) = r(A)\}$ to be the minimum number of transitions of any least resistance path in the set $A \subset \mathcal{A}(x, B, W)$.

Fast Theorem: *There are constants* $D_1(r), D_2(k, r)$ *with* $C \leq D_1(r), D_2(k, r) < \infty$ *such that* $C^{t(A)} \epsilon^{r(A)} \leq P_\epsilon(A|x) \leq D_1(r(A)) \epsilon^{r(A)}$*; and* $E[t^k(a)|x, A] \leq D_2(k, r(A))/C^{t(A)}$.

Elements of Proof

- the lower probability bound is fairly obvious from considering a least resistance path of shortest length
- the upper bound must take account of the fact that there are generally many more paths that have greater than least resistance than paths of least resistance so the key is to show that longer paths are a lot less likely than shorter paths
- we show that longer paths are constructed from shorter paths by inserting zero or low resistance loops
- these loops are not very likely because the comprehensive set W will probably get hit instead, and this can be used to show that the probability of longer paths declines exponentially
- since longer paths are a lot less likely than shorter, we also get an estimate of their length (i.e. short)

Least Resistance Paths are Most Likely

Main Corollary: *Let* $A = \{a | r(a) = r(\mathcal{A}(x, B, W))\}$ *denote the least resistance paths in* $\mathcal{A}(x, B, W) \neq \emptyset$. *Then*

$$\lim_{\epsilon \to 0} \frac{P_\epsilon(A|x)}{P_\epsilon(\mathcal{A}(x, B, W) \backslash A|x)} = \infty$$

applying to the illustrative example yields facts (1) and (2) concerning transitions between the ergodic sets

Minor Corollary: *Let* $A = \{a | r(a) = r(\mathcal{A}(x, B, W))\}$ *and* $a \in A$. *Then*
$\frac{P_\epsilon(a|x)}{P_\epsilon(A|x)} \geq C^{t(a)} / D_1(r(A))$

Transitions Between Irreducible Classes

an initial point $x \in Z$ with $\Omega(x) \neq \emptyset$ a forbidden set $W \subseteq Z$ and a target set $B \subseteq W$

direct routes from x to B with forbidden set W not allowed to pass through all points in $\Omega(x)$

relax that restriction, and consider routes which are allowed to linger freely inside $\Omega(x)$ so $W \cap \Omega(x) = \emptyset$ (so W cannot be comprehensive)

instead assume that W is *quasi-comprehensive*: contains at least one point from every irreducible class except for $\Omega(x)$

paths $\boldsymbol{A}(x, B, W)$ from x to B with forbidden set W which have positive probability for $\epsilon > 0$ called *quasi-direct routes*

Ellison observes that being able to pass through every point in an irreducible class may have a profound impact on the nature of the paths

Main Result and Setup

will show that before leaving $\Omega(x)$ for good, quasi-direct routes spend most of the time within $\Omega(x)$

assume the set $\boldsymbol{A}(x, B, W)$ is non-empty

interested in the structure of the paths, in particular: which paths in $\boldsymbol{A}(x, B, W)$ are most likely, what do these paths look like, and how long are they?

Decomposition of Quasi-Direct Paths

a a path in $\boldsymbol{A}(x, B, W)$: two distinct parts, the initial wandering in or near $\Omega(x)$ and the final crossing to B, or: returning to x a number of times before leaving x to hit B without returning

$\boldsymbol{A}^-$ set of paths that begin and end at x and do not touch W

$\boldsymbol{A}^+$ be the routes from x to B that do not touch W nor x in between - that is the direct routes to B with forbidden set $W \cup \{x\}$; since W is quasi-comprehensive $W \cup \{x\}$ is comprehensive, so these are indeed direct routes

we have the unique decomposition of a into the *equilibrium path* $a^- \in \boldsymbol{A}^-$ and the *exit path* $a^+ \in \boldsymbol{A}^+$.

Structure of Equilibrium Paths

a path $a \in \boldsymbol{A}^-$ can be decomposed into loops that begin and end at x but do not hit x

let $\mathcal{A}^0 = \mathcal{A}(x, \{x\}, W \cup \{x\})$ be the direct paths from x to $B = \{x\}$ avoiding the comprehensive set consisting of the quasi-comprehensive set W plus $\{x\}$ itself

paths in $\boldsymbol{A}^-$ are exactly sequences $a_1, a_2, \ldots, a_n$ such that $a_i \in \mathcal{A}^0$. We write $n(a)$ for the number of loops of a.

Peak Resistance

So: any path $a \in \boldsymbol{A}(x, B, W)$ has a unique decomposition $a_1, a_2, \cdots, a_n, a^+$ where the $a_i \in \mathcal{A}^0$ are the loops in $\mathcal{A}^0$ and $a^+ \in \boldsymbol{A}^+$ is the exit path to B.

the *equilibrium resistance* $\rho^-(a) = \max r(a_i)$

the *exit resistance* $\rho^+(a) = r(a^+)$

the *peak resistance* $\rho(a) = \max\{\rho^-(a), \rho^+(a)\}$

for $A \subseteq \boldsymbol{A}(x, B, W)$ the *least peak resistance* $\rho(A) = \min_{a \in A} \rho(a)$

Least Peak Resistance and Exit Resistance

The first thing to understand is that least peak resistance paths

are also least exit resistance paths:

Least Peak Resistance Theorem:
$\{a|\rho(a) = \rho(\boldsymbol{A}(x, B, W))\} \subseteq \{a|\rho^+(a) = \rho(\boldsymbol{A}(x, B, W))\}$

why? There was no reason to incur the extra resistance before leaving
$\Omega(x)$ just go right there

Express Exits

a least peak resistance path has two parts: a^- and a^+ which is a least resistance direct route from x to B with forbidden set $W \cup \{x\}$

some points $y \in \Omega(x)$ may support lower resistance direct routes to B with forbidden set $W \cup \Omega(x)$, that is, they are more likely to get there without returning to $\Omega(x)$: these are the *express exits*

a least peak resistance path from x to B must leave $\Omega(x)$ through an express exit

> since the express exit is also in $\Omega(x)$ it can be reached from x with zero resistance. Hence to leave $\Omega(x)$ through any other exit would incur higher resistance.

Likely Quasi-Direct Paths

Likely Quasi-Direct Path Theorem: Let $A = \{a | \rho(a) = \rho(\boldsymbol{A}(x, B, W))\}$ denote the least peak resistance paths in $\boldsymbol{A}(x, B, W) \neq \emptyset$. Then

$$\lim_{\epsilon \to 0} \frac{P_\epsilon(A|x)}{P_\epsilon(\boldsymbol{A}(x, B, W) \backslash A|x)} = \infty$$

Where Do Least Peak Resistance Path End?

Theorem not only tells us the most likely route from $\Omega(x)$ to $\Omega\backslash\Omega(x)$, by implication it tells us where we are likely to end up in $\Omega\backslash\Omega(x)$.

Let Ω_ρ irreducible classes in $\Omega\backslash\Omega(x)$ that are directly reachable from some point $y \in \Omega(x)$ with least resistance ρ

$\Omega_{-\rho} = \Omega\backslash(\Omega_\rho \cup \Omega(x))$

$P_\epsilon(\Omega_i|x)$ denote the probability starting at x the first arrival at $\Omega\backslash\Omega(x)$ is in Ω_i for $i = \rho, -\rho$.

Corollary: $\lim_{\epsilon\to 0} \frac{P_\epsilon(\Omega_\rho|x)}{P_\epsilon(\Omega_{-\rho}|x)} = \infty$

Equilibrium Paths

$\boldsymbol{A}^-$ are equilibrium paths: transition paths in the direct route case are short; paths that are allowed to remain in $\Omega(x)$ we have the opposite result: these paths are quite long

the only case of interest is where we reach B with probability one, that is $P_\epsilon(\boldsymbol{A}(x, B, W)|x) = 1$, so now assume that $B = W$

Recall that $a^- \in \boldsymbol{A}^-$ is a sequence $a_1, a_2, \ldots, a_n$ with $a_i \in \mathcal{A}^0$ loops at x. Now let $M(A^0)$ be the number of loops that lie in $A^0 \subseteq \mathcal{A}^0$ and let t_- be the amount of time along a^- spent outside of $\Omega(x)$.

Equilibrium Paths Theorem: If $B = W$ we have for some $C_1, D_3 > 0$

$$1 + C_1 \epsilon^{-r(\boldsymbol{A}^+)} \leq E[t|x, \boldsymbol{A}(x, B, W)] \leq D_2(1,0)C^{-N} + D_3 C^{-2N} \epsilon^{-r(\boldsymbol{A}^+)}$$

and for $A^0 \subseteq \mathcal{A}^0$

$$C_1 C^{t(A^0)} \epsilon^{r(A^0) - r(\boldsymbol{A}^+)} \leq E[M(A^0)|x, \boldsymbol{A}^-] \leq D_1(r(A^0)) C^{-2N} \epsilon^{r(A^0) - r(\boldsymbol{A}^+)}.$$

Moreover, $\lim_{\epsilon \to 0} E[t_-/t|x, \boldsymbol{A}^-] = 0$.

What the Theorem Says

quasi-direct paths are long and return to x many times, and in between most of the time is spent within $\Omega(x)$. Moreover if there is some $a^0 \in \mathcal{A}^0$ with $r(a^0) < r(\boldsymbol{A}^+)$ then the amount of time spent outside of $\Omega(x)$ is large in an absolute sense

gives assertions 3 and 4 in the example

The Big Picture

how much time do we spend in the various places?

assume now that P_ϵ is ergodic for $\epsilon > 0$

denote by μ_ϵ the unique ergodic distribution of the process.

Dynamics Within an Ergodic Class

Theorem: *let* A_1 *and* A_2 *be any collections of paths of bounded length and for which* $P_0(A_2|x) > 0$. *Then* $\lim_{\epsilon \to 0} \frac{P_\epsilon(A_1|x)}{P_\epsilon(A_2|x)} = \frac{P_0(A_1|x)}{P_0(A_2|x)}$.

if we restrict the state space to $\Omega(x)$ then P_0 is an ergodic Markov process on that space, so has a unique and strictly positive ergodic distribution $\overline{\mu}_0(y)$

Theorem: $y \in \Omega(x)$ *then* $\lim_{\epsilon \to 0} \frac{\mu_\epsilon(x)}{\mu_\epsilon(y)} = \frac{\overline{\mu}_0(x)}{\overline{\mu}_0(y)}$.

In the Basin

$r(\Omega(x),\Omega(y))$ is the least resistance of any direct path from x to the target $\Omega(y)$ with forbidden set $W = \{x\} \cup \Omega \setminus \Omega(x)$ - independent of the particular starting point x in $\Omega(x)$

W be all the irreducible classes, $r(\Omega(x)) = \min_{\Omega(y)\in W} r(\Omega(x),\Omega(y))$

Theorem: *Allowing that $\Omega(x)$ may be empty, if $A = \mathcal{A}(x,\{y\},\{x\}\cup\{y\}\cup(\Omega\backslash\Omega(x))$ are the direct routes from x to y with forbidden set $\{\mathrm{x}\}\cup\{y\}\cup(\Omega\backslash\Omega(x))$ then*

$$\mu_\epsilon(y) \geq \mu_\epsilon(x) C^N \epsilon^{r(A)}.$$

There is also a constant D_4 such that if $x \in \Omega(x)$ and there is a zero resistance path from y to x then also

$$\mu_\epsilon(y) \leq \mu_\epsilon(x) D_4 \epsilon^{\min\{r(A),r(\Omega(x))\}}.$$

In the Inner Basin

the *inner basin* of $\Omega(x)$ is set of points y that have zero resistance of reaching $\Omega(x)$ and in addition have resistance less than or equal to $r(\Omega(x))$ of being reached from x along a direct route with forbidden set $\{x\} \cup \{y\} \cup (\Omega \backslash \Omega(x))$

for the inner basin, the bounds in the theorem are tight

Long Run Ergodic Probabilities

start at x and go to $\Omega(y)$

long time before $\Omega(z)$ mostly spent in $\Omega(y)$

eventually move quickly – and directly – to the next $\Omega(z)$ most likely an irreducible set that has least exit resistance from $\Omega(y)$

hence a sequence of irreducible sets $\Omega(x_i)$ connected by least exit resistances. Since the

set W of irreducible classes in P_0 is finite, eventually this sequence must have a loop

more generally a *circuit* $\Phi \subseteq \Psi$ of a set of points on which a resistance function $r(\psi, \phi)$ is defined for each pair $\phi_1, \psi \in \Phi$ there is a path $\phi_1, \phi_2, \ldots, \phi_n \in \Phi \text{ with } \phi_n = \psi$ such that for $\tau = 2, 3, \ldots n$ we have $r(\phi_{\tau-1}, \phi_\tau) = r(\phi_{\tau-1})$ where $r(\psi) = \min_{\phi \in \Psi} r(\psi, \phi)$ is the *least resistance*

What Happens in Circuits?

once we reach a circuit, we remain within the circuit for a long time before going to another circuit

since we stay in ψ roughly $\epsilon^{-r(\psi)}$ periods before moving to another irreducible class in the circuit, we expect that the amount of time we spend at ψ is roughly $\epsilon^{r(\phi)-r(\psi)}$ as long as the amount of time we spend at ϕ

Same Circuit Theorem: *If the irreducible classes* $\Omega(x)$ *and* $\Omega(y)$ *are in the same circuit then*

$$\frac{C^N}{N^{N-1}D^N}\epsilon^{r(\Omega(y))-r(\Omega(x))} \leq \frac{\mu_\epsilon(x)}{\mu_\epsilon(y)} \leq \frac{N^{N-1}D^N}{C^N}\epsilon^{r(\Omega(y))-r(\Omega(x))}.$$

How Do We Get Out of a Circuit?

Expected length of any visit to ψ is $1/\epsilon^{r(\psi)}$.

probability of going to a fixed ϕ not in the circuit of order $\epsilon^{r(\psi,\phi)}$

hence probability of going to ϕ during a visit to ψ of order $(1/\epsilon^{r(\psi)})\epsilon^{r(\psi,\phi)}$

for this to happen we should visit ψ roughly k_ψ where $k_\psi(1/\epsilon^{r(\psi)})\epsilon^{r(\psi,\phi)} = 1$, that is $k_\psi = 1/\epsilon^{r(\psi,\phi)-r(\psi)}$

the modified resistance from ψ to ϕ is $R(\psi,\phi) = r(\psi,\phi) - r(\psi)$

the number of visits is least for the ψ which has minimum $R(\psi,\phi)$ over $\psi \in \Phi$

so next we try to form circuits of circuits using modified resistance as the measure of the cost of going from one circuit to another

Construction of the Reverse Filtration

a class of reverse filtrations with resistances over the set $\Psi^0 = W$ of irreducible sets for P_0

assume W has N_W elements, with $N_W \geq 2$

for $\psi, \phi \in W$ the resistance $r^0(\psi, \phi)$ is the least resistance of any direct path from $x \in \psi$ to the target ϕ with forbidden set $W = \{x\} \cup \Omega \setminus \Omega(x)$

starting with Ψ^{k-1} there is at least one non-trivial circuit, and that every singleton element is trivially a circuit so we can form a non-trivial partition of Ψ^{k-1} into circuits: denote this partition Ψ^k

define the modified resistance
$R^{k-1}(\psi^{k-1}, \phi^{k-1}) = r^{k-1}(\psi^{k-1}, \phi^{k-1}) - r^{k-1}(\psi^{k-1})$,

and the resistance function on Ψ^k by the least modified resistance:

$$r^k(\psi^k, \phi^k) = \min_{\psi^{k-1} \in \psi^k, \phi^{k-1} \in \phi^k} R^{k-1}(\psi^{k-1}, \phi^{k-1})$$

The Modified Radius

since each partition is non-trivial construction has at most $\mathrm{k}\leq N_W$ layers before the partition has a single element and the construction stops.

given the reverse filtration, for given $x \in \Omega(x)$ define $\psi^k(x)$ recursively by $x \in \psi^o(x), \psi^0(x) \in \psi^1(x) \ldots$

modified radius of $x \in \Omega(x)$ of order k is

$$\overline{R}^k(x) = \sum_{\kappa=0}^{k} r^\kappa(\psi^\kappa(x)).$$

The Ergodic Ratios

Theorem: *Let* k *be such that* $\psi^k(x) = \psi^k(y)$*; then*

$$\frac{C^N}{N^{N-2}D^N}\epsilon^{\overline{R}^{k-1}(y)-\overline{R}^{k-1}(x)} \leq \frac{\mu_\epsilon(x)}{\mu_\epsilon(y)} \leq \frac{N^{N-2}D^N}{C^N}\epsilon^{\overline{R}^{k-1}(y)-\overline{R}^{k-1}(x)}.$$

gives fifth assertion concerning the example

The Three Element Case

W has three elements

9 trees on 3 points, so the analysis by means of trees is already difficult

make the generic assumption that no two resistances or sums or differences of resistances are equal.

two cases:

- a single circuit
- one circuit consisting of two points, and a separate isolated point

single circuit is trivial

- relative ergodic resistances differences in least resistances between the three points
- stochastically stable state is the point with least least resistance

A Circuit and a Point

denote by ϕ_a, ϕ_b the two points on the circuit with ϕ_c the remaining point

without loss of generality $r(\phi_a) > r(\phi_b)$ so within the circuit ϕ_a is relatively more likely

since ϕ_a, ϕ_b are on the same circuit

$r(\phi_a, \phi_b) < r(\phi_a, \phi_c), r(\phi_b, \phi_a) < r(\phi_b, \phi_c)$

also implies

$r(\phi_a) = r(\phi_a, \phi_b), r(\phi_b) = r(\phi_b, \phi_a)$

Least Modified Resistances

$\psi_a = \{\phi_a, \phi_b\}$ be the circuit and $\psi_c = \phi_c$ the isolated point

$$r^1(\psi_c, \psi_a) = \min\{r(\phi_c, \phi_a) - r(\phi_c), r(\phi_c, \phi_b) - r(\phi_c)\} = 0$$

$$r^1(\psi_a, \psi_c) = \min\{r(\phi_a, \phi_c) - r(\phi_a), r(\phi_b, \phi_c) - r(\phi_b)\}$$

hence $\overline{R}^1(\phi_c) = r(\phi_c)$, i.e. the radius of ϕ_c while

$$\begin{aligned}\overline{R}^1(\phi_a) &= r(\phi_a) + \min\{r(\phi_a, \phi_c) - r(\phi_a), r(\phi_b, \phi_c) - r(\phi_b)\} \\ &= \min\{r(\phi_a, \phi_c), r(\phi_a) + r(\phi_b, \phi_c) - r(\phi_b)\}\end{aligned}$$

exactly what Ellison defines as the modified co-radius of ϕ_c.

relative ergodic resistance of ϕ_a over ϕ_c is $r(\phi_c) - \overline{R}^1(\phi_a)$

relative ergodic resistance of ϕ_b can be recovered from the relative ergodic resistance of ϕ_b over ϕ_a which is $r(\phi_a) - r(\phi_b)$.

Stochastic Stability

- ϕ_c is stochastically stable if and only if its radius $r(\phi_c)$ is greater than its co-radius $\overline{R}^1(\phi_a)$ (Ellison's sufficient condition)
- otherwise ϕ_a is stochastically stable

the entire ergodic picture comes down to computing three numbers: the radius and co-radius of ϕ_c and the difference between the radii of ϕ_a and ϕ_b.

Stationary Distribution

the stationary distribution is denoted $\mu(\epsilon)$

Theorem: *For* $\epsilon > 0$ *there is a unique* $\mu(\epsilon)$ *and it places positive weight on all states. As* $\epsilon \to 0$ *there is a unique limit* μ*. There exists an* a_0^* *such that if* $\max_{j, b_j = 1} a_j^O < a_0^*$ *then* μ *places positive weight on all states. If* $\max_{j, b_j = 1} a_j^O \geq a_0^*$ *then* μ *places weight only on hegemonic states* j *that have maximal equilibrium state power*

$$a_j^O = \max_{j', b_{j'} = 1} a_{j'}^O$$

generalizes result from two society birth-death example: with strong outsiders there is no tendency towards hegemony, with weak outsiders there is and it is a hegemony of the strongest **equilibrium**

Some Facts About Hegemony

- China: 2,234 years from 221 BCE – hegemony 72% of time, five interregna
- Egypt: 1,617 years from 2686 BCE - hegemony 87% of time, two interregna
- Persia: 1,201 years from 550 BCE - hegemony 84% of time, two interregna
- England: 947 years from 1066 CE - hegemony 100% of time
- Roman Empire: 422 years from 27 BCE - hegemony 100% of time
- Eastern Roman Empire: 429 years from 395 CE – 100%
- Caliphate: 444 years from 814 CE – 100%
- Ottoman Empire: 304 years from 1517 CE – 100%

Remark: in 0 CE 90% of world population in Eurasia/North Africa

Exceptions

- India
- continental Europe post Roman Empire

evolutionary theory: more outside influence, less hegemony

- Europe: Scandinavia 5%, England 8%
- India: Central Asia 5%
- China: Mongolia less than 0.5%

Hegemonic Transitions

assume hereafter that ϵ is small and look at transitions between different hegemonic states

the *fall* of a hegemony is time at which the hegemony is lost and another hegemony is reached without returning to the original hegemony

Length of Transitions

Theorem: *The expected length of time for a hegemony to be reached is bounded independent of ϵ. When $\max_{j, b_j = 1} a_j^O \geq a_0^*$ the expected amount of time before hegemony falls grows without bound as $\epsilon \to 0$.*

Should not expect much difference in the time between hegemonies in different regions – the regions where hegemony is more common should have longer lasting hegemonies, but not less time for hegemony to be reached.

Historical Facts About Transitions

average time to hegemony from end of previous hegemony

- China (220 CE to present): 153 years
- Egypt (2160 BCE to 1069 BCE): 102 years
- Persia (550 BCE to 651 CE): 145 years
- Western Europe (295 CE to present): 366 years
- India (320 CE to present): 209 years

Strong Hegemonies

a hegemony is *strong* if it has positive resistance when it has lost a single unit of land

Theorem: *As* $\epsilon \to 0$ *the number of times a strong hegemony will lose land before it falls grows without bound.*

true in China during the period during which we have good data during the century prior to the fall of the Ching hegemony in 1911

many failed attempts at revolution, most notably

- Boxer rebellion in 1899
- Dungan revolt in 1862 – lasted 15 years and involved loss of control in a number of provinces

in each case hegemony was restored.

Types of Transitions

Theorem: *As $\epsilon \to 0$ the probability that the path between hegemonies is a least resistance path approaches one.*

The next step is to analyze what least resistance paths between hegemonies look like.

Zealots

assume $\max_{j,b_j=1} a_j^O \geq a_0^*$ and ϵ small (so hegemonies commonplace)

assume $\max_j a_j^O > \max_{j,b_j=1} a_j^O$

k that achieves the max called *zealots*

- zealots by definition do not satisfy incentive constraints
- the “ethos of the warrior/revolutionary”
- could be deviant preferences
- essential point is that while they are strong, zealots are not stable – they do not form societies that last

assume hereafter that there are zealots

Role of Zealots in Transitions

Theorem: *As* $\epsilon \to 0$ *when a hegemony loses an amount of land* $L_- > 1$ *the land with probability approaching one the land is taken by zealots and the process is monotone (zealots never lose any land along the path)*

Facts About Zealots

groups that overcame strong hegemonies (where we have data)

- Sun Yat Sen's revolutionaries
- Mongolian groups that overcame other Chinese dynasties
- Huns led by Attila

All have been willing to sacrifice material comfort for the cause (institutional change or conquest). This idealism rarely lasted even a generation.

All have been well-organized and efficient

Revolts and invasions against strong hegemonies are generally either repressed and or unchecked and succeed.

Least ResistancePaths

- begin with zealots gaining land
- after a threshold is reached there is a warring states period in which the hegemony no longer has positive resistance

we refer to the beginning of the warring states period as the *collapse* of the hegemony

Theorem: *The expected length of time for a hegemony to collapse is bounded independent of* ϵ*.*

it should not depend on the duration of the hegemony that collapsed

Where we have recent and fairly accurate data collapses brutally fast:

- Ching hegemony established in 1644 CE (and institutions that lasted since 605 CE) swept permanently away in 1911 in well less than a year, and less time even than the fall of the very short lived hegemonies established by Napoleon or Hitler.

Transition to Hegemony

Theorem: *With zealots the probability of reaching any particular hegemony is bounded away from zero independent of ϵ.*

the *least resistance* of a hegemony is the resistance of the least resistance path to another hegemony: it is a measure of the strength of the hegemonic institutions relative to outside forces

no particular tendency to reach any type of hegemony, weak or strong

Facts About the Emergence of Hegemonic Institutions

short lived hegemonies

- Alexander – weak institutions
- Napolean – strong outside forces
- Hitler – strong outside forces
- Soviet Union – weak institutions and strong outside forces

long lived hegemonies where zealots initiated a hegemony

- various Mongol invaders of China – adopted Chinese institutions

the theory says following the warring states period anything can happen: and it does

Conclusion

The theory says that if we start from the observation that institutions tend to evolve through conflict between societies, rather than, say, through peaceful competition for resources, then other things should also be true:

- persistent hegemony and extractiveness in circumstances where outside forces are weak
- time to hegemony largely independent of circumstances
- fall of strong hegemonies due to “perfect storm” following many failed revolts