Econ 506A (2008)
Topics in Advanced Theory I
GAME THEORY

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Lattices, Sublattices, Fixed Points, and Supermodularity
Lattices
Partial orders

A binary relation $\geq$ on a set $X$ is a partial order if it is:

1. Reflexive: $x \geq x$ for all $x \in X$.

2. Transitive: $x \geq y, y \geq z \Rightarrow x \geq z$.

3. Antisymmetric: $x \geq y, y \geq x \Rightarrow x = y$.

Two elements $x, y$ in a partially ordered set are ordered if $x \geq y$ or $y \geq x$. A partial order is a chain (total/strict/linear order) if any two elements are ordered.

We write $x \leq y$ to denote $y \geq x$. We write $x > y$ to denote $x \geq y$ and not $x \leq y$ (similarly for $x < y$).
Lattices

A partially ordered set \((X, \geq)\) is a **lattice** if for any \(x, y \in X\), there exist elements \(x \lor y, x \land y \in X\) such that:

1. \(x \lor y \geq x, y\) and \(\forall z \in X\) s.t. \(z \geq x, y\) we have \(z \geq x \lor y\).

2. \(x \land y \leq x, y\) and \(\forall z \in X\) s.t. \(z \leq x, y\) we have \(z \leq x \land y\).

\(x \lor y\) is called the **join** and \(x \land y\) is called the **meet**.
Examples

Euclidean space: $\mathbb{R}^n$ is a lattice under the partial order:

$$x \leq y \iff x_i \leq y_i \text{ for all } i = 1, \ldots, n.$$  

The meet and join of $x, y \in \mathbb{R}^n$ are given by:

$$x \land y = \left( \min \{x_1, y_1\}, \min \{x_2, y_2\}, \ldots, \min \{x_n, y_n\} \right),$$

$$x \lor y = \left( \max \{x_1, y_1\}, \max \{x_2, y_2\}, \ldots, \max \{x_n, y_n\} \right).$$

When $n = 1$, $\mathbb{R}$ is a chain.

Power set: Given a set $X$, the power set $2^X$ which denotes the set of all subsets of $X$ is a lattice under the partial order $\supset$. The meet and join of $A, B \in 2^X$ are given by:

$$A \land B = A \cap B \text{ and } A \lor B = A \cup B.$$
Examples (continued)

**General Product Lattices:** Given a collection of lattices \((X_\alpha, \geq_\alpha)_{\alpha \in I}\), define the binary relation \(\geq\) on the Cartesian product \(X = \prod_{\alpha \in I} X_\alpha\) by:

\[
x \geq y \iff \forall \alpha \in I : x_\alpha \geq_\alpha y_\alpha.
\]

Then, \((X, \geq)\) is a lattice, \(x \land y = (x_\alpha \land y_\alpha)_{\alpha \in I}\), and \(x \lor y = (x_\alpha \lor y_\alpha)_{\alpha \in I}\). \((X, \geq)\) is called the **product lattice**.
Tarski’s Fixed Point Theorem
Complete lattices

Given a partially ordered set \((X, \geq)\) and \(A \subset X\):

1. An **upper bound** of \(A\) is \(\bar{x} \in X\) s.t. \(\forall x \in A : \bar{x} \geq x\).

2. A **least upper bound/supremum** of \(A\) is an upper bound \(x^* \in X\) of \(A\) s.t. \(\bar{x} \geq x^*\) for any upper bound \(\bar{x}\) of \(A\).

When supremum of \(A\) exists, it is unique, and denoted by \(\sup_X(A)\). Lower bound and \(\inf_X(A)\) are defined similarly.

Note: Any finite subset of a lattice has a supremum and an infimumum.

A lattice \((X, \geq)\) is **complete** if for all nonempty \(A \subset X\), \(\inf_X(A)\) and \(\sup_X(A)\) exist.
Tarski’s Fixed Point Theorem

Given a lattice \((X, \geq)\) and a function \(f : X \to X\):

- \(f\) is isotone if \(x \geq y \Rightarrow f(x) \geq f(y)\).
- \(x^* \in X\) is a fixed point of \(f\) if \(f(x^*) = x^*\).

**Theorem** (Tarski (1955)) Let \((X, \geq)\) be a complete lattice and let \(f : X \to X\) be an isotone function. Then,

1. The set of fixed points of \(f\) is nonempty, and
   (a) \(\text{sup}_X\{x \in X : f(x) \geq x\}\) is the largest fixed point.
   (b) \(\text{inf}_X\{x \in X : f(x) \leq x\}\) is the smallest fixed point.

2. The set of fixed points of \(f\) is a nonempty complete lattice.
Constructive Proof of (1a) when $X$ is Finite

Let $\bar{x} = \sup_X X$ and $x^n = f^n(\bar{x})$. Then

$$x^0 \geq x^1 \geq x^2 \geq \ldots$$

By finiteness of $X$, there exists $n^*$ such that $x^{n^*} = x^{n^*+1}$.

Let $\bar{x}^* := x^{n^*}$. Note that:

- $\bar{x}^*$ is a fixed point of $f$.
- $f(x) \geq x \Rightarrow \bar{x}^* \geq x$.

So, $\bar{x}^* = \sup_X \{x \in X : f(x) \geq x\}$ is the largest fixed point of $f$. 
Sublattices
Let \((X, \geq)\) be a lattice. We extend the partial order \(\geq\) to all subsets of \(S, T \subset X\) using the induced set ordering:

\[
S \geq T \iff \forall x \in S, \forall y \in T : x \lor y \in S \text{ and } x \land y \in T.
\]

The subset \(S \subset X\) is a sublattice of \(X\) if \(S \geq S\), or more explicitly if for all \(x, y \in S\):

\[
x \land y, x \lor y \in S.
\]

**Notes:** The meet and join operations of a sublattice \(S\) are the same as the meet and join operations of \((X, \geq)\). That is not true for all \(Y \subset X\) such that \((Y, \geq|_Y)\) is a lattice.

Arbitrary intersections of sublattices of \(X\) is a sublattice of \(X\). This is not true for subsets of \(X\) that are lattices.
Characterization of the Sublattices of a Finite Product of Lattices

A sublattice of a product lattice can be characterized by a collection of pairwise restrictions:

**Theorem** Let \((X, \geq)\) be the product of finitely many lattices \((X_1, \geq_1), \ldots, (X_n, \geq_n)\). Let \(S\) be an arbitrary subset of \(X\) and for any \(i, j\) with \(i \neq j\), define:

\[
\tilde{S}_{ij} = \{(y_i, y_j) \in X_i \times X_j : \exists x \in S \text{ such that } y_i = x_i \text{ and } y_j = x_j\}.
\]

and \(S_{ij} = \tilde{S}_{ij} \times X_{-ij}\).

Then, \(S\) is a sublattice of \(X\) if and only if

1. \(\tilde{S}_{ij}\) is a sublattice of \(X_i \times X_j\) for \(i \neq j\).

2. \(S = \cap_{i \neq j} S_{ij}\).
Supermodularity
Given a lattice \((X, \geq)\), a function \(f : X \to \mathbb{R}\) is **supermodular** if for all \(x, y \in X\):

\[
f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).
\]

\(f\) is **submodular** if \(-f\) is supermodular.

Note that if all pairs are ordered under \(\geq\) (e.g. \(\mathbb{R}\)), then all functions are supermodular and submodular!
Increasing Differences

Let \( f : X \times Y \to \mathbb{R} \) where \( X \) and \( Y \) are two lattices. Then, \( f \) has **increasing differences** if for all \( x, x' \in X \) with \( x \geq x' \), the difference \( f(x, y) - f(x', y) \) is nondecreasing in \( y \).

(Definition does not change if coordinates are reversed.)

Let \( (X, \geq) \) be a product of finitely many lattices \( (X_i, \geq_i)^n_{i=1} \) and \( f : X \to \mathbb{R} \). Then, \( f \) has **increasing differences** if \( f(\cdot, \cdot, x_{-ij}) : X_i \times X_j \to \mathbb{R} \) has increasing differences for all \( i, j \) with \( i \neq j \) and \( x_{-ij} \in X_{-ij} \).

**Lemma** Let \( (X, \geq) \) be a product of finitely many lattices \( (X_i, \geq_i)^n_{i=1} \) and \( f : X \to \mathbb{R} \). If \( f \) is has increasing differences, then \( \forall i : \)

\[
x \geq y \Rightarrow f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i}).
\]
Supermodularity on Finite Products of Lattices

Let \((X, \geq)\) be a product of finitely many lattices \((X_i, \geq_i)_{i=1}^{n}\) and \(f : X \to \mathbb{R}\). Then, \(f\) is coordinatewise supermodular if for all \(i\) and \(x_i \in X_i\), \(f(\cdot, x_i) : X_i \to \mathbb{R}\) is supermodular.

**Theorem** Let \((X, \geq)\) be a product of finitely many lattices \((X_i, \geq_i)_{i=1}^{n}\) and \(f : X \to \mathbb{R}\). Then, \(f\) is supermodular if and only if \(f\) is coordinatewise supermodular and has increasing differences.

If \(X\) is a product of chains (e.g. \(\mathbb{R}^n\)), then any function \(f\) satisfies coordinatewise supermodularity, implying:

**Corollary** If \(X\) is a product of chains, then \(f : X \to \mathbb{R}\) is supermodular if and only if it has increasing differences.
Supermodularity on $\mathbb{R}^n$

**Lemma** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Then, $f$ has increasing differences if and only if
\[
\frac{d^2f}{dx_1dx_2} \geq 0.
\]

**Corollary** Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then, $f$ is supermodular if and only if it has nonnegative mixed second order derivatives, i.e.
\[
\frac{d^2f}{dx_idx_j} \geq 0
\]
for all $i \neq j$. 
Topkis’ Monotonicity Theorem
Theorem  Let $X$ be a lattice, $T$ be a partially ordered set, $f : X \times T \to \mathbb{R}$, and $S : T \to 2^X \setminus \{\emptyset\}$. Assume that $f(x,t)$ is supermodular in $x$ for each $t$, and it has increasing differences in $x$ and $t$ in the following sense:

$$f(x,t) - f(x',t) \geq f(x,t') - f(x',t')$$

whenever $x \geq x'$ and $t \geq t'$. Define $x^* : T \to 2^X$ by:

$$x^*(t) = \arg \max_{x \in S(t)} f(x,t).$$

If $t \geq t'$ and $S(t) \supseteq S(t')$, then $x^*(t) \supseteq x^*(t')$.

A function $f : X \to Y$, where $X$ and $Y$ are sets endowed with arbitrary binary relations, is isotone (or monotone) if $x \geq y \Rightarrow f(x) \geq f(y)$.

Corollary  If in the monotonicity theorem, $S : T \to 2^X \setminus \{\emptyset\}$ isotone, then $\forall t \in T$, $S(t)$ and $x^*(t)$ are sublattices of $X$. 