Econ 506A (2008)
Topics in Advanced Theory I
GAME THEORY

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Supermodular Games
Milgrom and Roberts (1990)
Supermodular Games

Players: \( N = \{1, \ldots, n\} \).

Pure strategies of \( i \): \( A_i \) is a compact sublattice of \( \mathbb{R}^{k_i} \). Then, \( x_i := \inf_{\mathbb{R}^{k_i}} A_i \) and \( \bar{x}_i := \sup_{\mathbb{R}^{k_i}} A_i \) exist & are in \( A_i \).

Payoffs: \( u_i : A \rightarrow \mathbb{R} \):

1. Upper semi-continuous in \( x_i \), continuous in \( x_{-i} \).
2. Supermodular in \( x_i \), increasing differences in \( x_i \) and \( x_{-i} \).

Strategic complementarities: When other players increase their choice variables, it becomes more profitable for player \( i \) to increase hers as well.

Notes: The following results apply (w/out any continuity cond.) when pure strategy sets are finite lattices. They also apply to general complete lattices if the continuity conditions are appropriately reformulated (see M&R).
Example: Linear Bertrand Oligopoly with Differentiated Products

\( x_i \in A_i = [0, M] \) denotes the price set by firm \( i \).

Demand for firm \( i \):

\[
d_i(x) = a_i - b_i^i x_i + \sum_{j \neq i} b_j^i x_j
\]

where \( a_i, b_i^i, b_j^i \geq 0 \) are constants. Profit of firm \( i \):

\[
u_i(x) = (x_i - c_i) d_i(x)
\]

Increasing differences:

\[
\frac{d^2 u_i}{dx_i dx_j} = b_j^i \geq 0 \text{ for } i \neq j
\]
Example: Partnership Game

\( N = \{1, 2\} \). Player 1 is the supplier of capital and player 2 is the supplier of labor: \( A_1 = [0, \bar{K}] \), \( A_2 = [0, \bar{L}] \).

Payoffs:

\[
u_1(K, L) = t\frac{K^\alpha L^\beta}{2} - K \quad \text{and} \quad u_2(K, L) = t\frac{K^\alpha L^\beta}{2} - L
\]

where \( t, \alpha, \beta > 0 \) are constants such that \( \alpha + \beta < 1 \) (decreasing returns to scale).

Increasing differences:

\[
\frac{d^2u_i}{dLdK} = \frac{\alpha \beta}{2K^{1-\alpha}L^{1-\beta}} > 0 \quad \text{for} \quad i = 1, 2.
\]

Also note that \( u_1 \) has increasing differences in \( t \) and \( K \); \( u_2 \) has increasing differences in \( t \) and \( L \).
Example: Cournot Duopoly

$x_i \in A_i = [0, M]$ denotes the quantity supplied by firm $i$.

Inverse demand (market clearing price):

$$P(x) = a - x_1 - x_2.$$  

where $a \geq 0$ is a constant. Profit of firm $i$:

$$u_i(x) = x_i[P(x) - c_i]$$

Decreasing differences!

$$\frac{d^2u_i}{dx_idx_j} = -1 \text{ for } i \neq j.$$ 

If we reverse the order on one of the players' pure strategy sets, then Cournot Duopoly becomes supermodular and we can apply the following results. (What about Cournot oligopoly with $n \geq 3$?)
Example: Diamond Search Model

$x_i \in A_i = [0, M]$ denotes the effort exerted by player $i$ to find a trading partner.

Probability of trade of player $i$ is $t \times g(\text{avg}(x_{-i})) \times x_i$, where $\text{avg}(x_{-i}) = \frac{1}{n-1} \sum_{j \neq i} x_j$, $g : [0, M] \to \mathbb{R}_+$ is nondecreasing, and $t > 0$ is constant. Payoff of $i$:

$$u_i(x) = tg(\text{avg}(x_{-i}))x_i - \frac{x_i^2}{2}$$

Increasing differences:

$$\frac{du_i}{dx_i} \text{ increasing in } x_j \text{ for } i \neq j$$

Also note that $u_i$ has increasing differences in $t$ and $x_i$. 
Best reply

**Lemma** Let \((N, A, u)\) be a supermodular game. Then,

1. \(\forall x \in A\), the pure strategy best replies of \(i\) to \(x_{-i}\):

\[
B_i(x) = \arg \max_{y_i \in A_i} u_i(y_i, x_{-i})
\]

is a nonempty compact sublattice of \(\mathbb{R}^{k_i}\).

2. The correspondence \(B_i\) is isotone and upper semi-continuous.

3. The extremal best replies

\[
\bar{B}_i(x) := \sup B_i(x) \quad \text{and} \quad \underline{B}_i(x) := \inf B_i(x)
\]

exist and are isotone.
Rationalizability and Nash Equilibria

**Theorem** Given a supermodular game \((N, A, u)\), the limits:

\[
\bar{z} = \lim_{k \to \infty} \bar{B}^k(\bar{x}) \quad \text{and} \quad z = \lim_{k \to \infty} B^k(x)
\]
exist and are pure strategy Nash equilibria.

Furthermore, rationalizable strategy profiles are a subset of \([z, \bar{z}] = \{x \in A : z \leq x \leq \bar{z}\}\).

**Lemma** Let \(\bar{x}^k := \bar{B}^k(\bar{x})\).

1. If \(x_i \not\in \bar{B}_i(\bar{x})\), then \(x_i\) is strictly dominated by \(x_i \wedge \bar{B}_i(\bar{x})\).

2. If \(x_i \not\in \bar{x}^k + 1\), then \(u_i(x_i \wedge \bar{x}^k + 1, y_{-i}) > u_i(x_i, y_{-i}) \forall y \leq \bar{x}^k\).
Corollaries

In the following fix, a supermodular game \((N, A, u)\).

**Corollary** Each player \(i\) has a minimum and a maximum rationalizable strategy given by \(z_i\) and \(\bar{z}_i\).

**Corollary** There exists a pure strategy Nash equilibrium.

**Corollary** If there is a unique Nash equilibrium, then there is a unique rationalizable strategy profile.

**Corollary** Suppose that the game is symmetric (unchanged by permutations of player indices). Then, it has a symmetric pure strategy Nash equilibrium. Furthermore, if it has unique symmetric Nash equilibrium, then there is a unique rationalizable strategy profile.
Theorem Let $T$ be a partially ordered set. Suppose that $(N, A, u(\cdot, t))_{t \in T}$ is a parametrized family of supermodular games such that:

- $u_i$ has increasing differences in $x_i$ and $t$.

Then, the largest and smallest Nash equilibria $\bar{z}(t)$ and $\underline{z}(t)$ are isotone in $t$. 

Comparative statics