

Econ 504 (2008) Problem Set #2

1. (First-Price Auction) Consider the the first-price auction with private values, where valuations are distributed i.i.d. with density $f(x) = 3x^2$ on $[0, 1]$.
 - (a) What is the BNE bid function?
 - (b) What are the expected revenues of the seller and the expected equilibrium payoff of a bidder with valuation $x \in [0, 1]$.
 - (c) What happens in (a) and (b) as $n \rightarrow \infty$?
2. (Second-Price Auction) Find a BNE of the second-price auction with private values, where bidder 1 always wins the object without making any payments.
3. (Average-Price Auction) Consider the following auction with private values. There are two bidders whose valuations are i.i.d. uniformly distributed on the interval $[0, 1]$. They simultaneously submit *nonnegative* bids and the highest bidder wins the object. If they make the same bid then they both win the object with equal probability. The winner pays the *average* of the two bids. The bidder who does not win makes no payments and gets a reservation utility of zero. Conjecture that there is a symmetric BNE with a differentiable bid function $b: [0, 1] \rightarrow \mathbb{R}$ such that $b'(x) > 0$ for any $x \in [0, 1]$.
 - (a) Write out the expected payoff of bidder i with valuation x when she bids b_i , given that the other player bids according to $b(\cdot)$.
 - (b) Find the necessary first order condition for b_i to maximize i 's expected payoff. After you derive the first order condition, simplify it conjecturing that $b_i = b(x)$ is the optimal bid for bidder i with valuation x .
 - (c) Solve for the bid function $b(\cdot)$.
 - (d) Compute the equilibrium expected payoffs of a bidder with valuation x . Compute the equilibrium expected revenues of the seller. How do they compare to the first and second price auctions?
 - (e) Show that the bid function $b(\cdot)$ you derived above from the necessary first order conditions, is in fact optimal.

4. (All-Pay Auction) In the “all-pay” auction, the highest bidder wins the object, every bidder (even if she does not win the object!) pays her bid. If there are multiple highest bidders, then one of them is chosen at random to win the object. Formally:

$$u_i(b_1, \dots, b_n; x_1, \dots, x_n) = \begin{cases} \frac{x_i}{|\{j \in N: b_j = b^1\}|} - b_i & \text{if } b_i = b^1 \\ -b_i & \text{otherwise.} \end{cases}$$

where b^1 denotes the highest bid among b_1, \dots, b_n . As usual, the bidders' valuations are i.i.d. with a continuous and strictly positive density f and c.d.f. F . Conjecture that there is a symmetric BNE with a differentiable bid function $b: [0, 1] \rightarrow \mathbb{R}_+$ such that $b'(x) > 0$ for any $x \in [0, 1]$.¹

- (a) Write out the expected payoff of bidder i with valuation x when she bids b_i , given that the others bid according to $b(\cdot)$.
 - (b) Find the necessary first order condition for b_i to maximize i 's expected payoff. After you derive the first order condition, simplify it conjecturing that $b_i = b(x)$ is the optimal bid for bidder i with valuation x .
 - (c) Solve for the bid function $b(\cdot)$. (Explain why the integration constant must be zero.)
 - (d) Compute the equilibrium expected payoffs of a bidder with valuation x . Compute the equilibrium expected revenues of the seller. How do they compare to the first and second price auctions?
 - (e) Show that the bid function $b(\cdot)$ you derived above from the necessary first order conditions, is in fact optimal.
5. (Double Auction) A seller s of a single indivisible object and a buyer b simultaneously choose prices $p_s, p_b \in [0, 1]$. If $p_b < p_s$, then there is no trade. If $p_b \geq p_s$, then they trade at price $p = \frac{p_b + p_s}{2}$. Valuations are private information x_b, x_s i.i.d.

¹Note that, just like in the previous question, here we are ruling out negative bids, which could be interpreted as monetary transfers from the seller to the bidder. The reason is that these two auctions have no BNE when negative bids are allowed for (why?), a concern we did not have for the type of auctions studied in class.

uniform on $[0, 1]$. Therefore payoffs are given by:

$$\begin{aligned}
 u_b(p_b, p_s; x_b) &= \begin{cases} x_b - \frac{p_b + p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases} \\
 u_s(p_b, p_s; x_s) &= \begin{cases} \frac{p_b + p_s}{2} - x_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

- (a) Show that there is a BNE where trade never occurs.
- (b) Show that for any $c \in [0, 1]$, there is a BNE where trade occurs if and only if $x_b \geq c \geq x_s$.
- (c) Find a BNE equilibrium in linear strategies, i.e.:

$$\begin{aligned}
 p_b(x_b) &= a_b + c_b x_b \\
 p_s(x_s) &= a_s + c_s x_s.
 \end{aligned}$$

for some $a_s, a_b, c_b, c_s > 0$.

6. (Interdependent Valuations) Suppose Y, Z_1, Z_2 are i.i.d. uniformly distributed on $[0, 1]$. Two bidders receive private signals X_1 and X_2 where $X_i = Y + Z_i$ for $i = 1, 2$. The valuation of bidder i for the object is given by $u_i(X_1, X_2) = \frac{1}{2}(X_1 + X_2)$.
 - (a) (Optional, but good exercise) Show that X_1 and X_2 are affiliated.
 - (b) Find the equilibrium bid function in the second-price auction. What are the seller's expected revenues?
 - (c) Find the equilibrium bid function in the first-price auction. What are the seller's expected revenues?
 - (d) Find the equilibrium bid function and the seller's expected revenues in the English auction.²
7. (Allocation of Multiple Indivisible Objects) Consider the allocation of three identical indivisible objects to three agents each of whom can consume multiple units. The types of each agent is given by $\Theta_i = \{(v_1, v_2, v_3) \in \mathbb{R}_+^3 \mid v_1 \leq v_2 \leq v_3\}$ where v_k denotes her valuation for k units. Consider the type profile:

²Hint: There is an easy answer to this part.

	θ_1	θ_2	θ_3
v_1	5	4	7
v_2	10	9	10
v_3	15	16	12

What is the allocation of the three goods and the transfers according to the pivotal VGC mechanism at the above type profile?

8. (Public Project) Consider the quasi-linear mechanism design framework. Let $X = \{x_0, x_1\}$ where x_1 corresponds to undertaking a public project and x_0 corresponds to not undertaking it. The net value of an individual $i \in N = \{1, \dots, n\}$ from not undertaking the project is zero, her value from undertaking the public project is $\theta_i \in \Theta_i \equiv [-1, +1]$. The value θ_i is the private information of individual i .
- Under the direct mechanism $(\Theta; f, 0)$ without transfers, is truth-telling a dominant strategy equilibrium? If you answer is yes explain why, otherwise give a profile of types where an individual has an incentive to misrepresent her type.
 - Derive the pivotal VGC mechanism.
 - Is there a budget-balanced VGC mechanism?
 - Suppose that each agent i has reservation utility $\underline{u}_i = 0$. Is there a VGC mechanism that requires no outside subsidies and is individually rational.
9. (Public Good) Consider a society $N = \{1, \dots, n\}$ who need to decide on the provision of a public good $y \in \mathbb{R}_+$. Each agent i 's quasi linear utility is given by $u_i(y, \theta_i) = \theta_i \sqrt{y} - \frac{1}{n}y$ where her type $\theta_i \in \Theta_i \equiv \mathbb{R}_+$ is her private information.
- Under the direct mechanism $(\Theta; f, 0)$ without transfers, is truth-telling a dominant strategy equilibrium? If you answer is yes explain why, otherwise give a profile of types where an individual has an incentive to misrepresent her type.
 - Derive the pivotal VGC mechanism.
 - Is there a budget-balanced VGC mechanism?
 - Suppose that each agent i has reservation utility $\underline{u}_i = 0$. Is there a VGC mechanism that requires no outside subsidies and is individually rational.