Econ 504 (2008) Problem Set #1

Warm up:

1. Do the following exercises from the first homework of John Nachbar’s Econ 504 (Spring 2007): 2(a), 4, and 7.

2. Do the following exercises from the Osborne-Rubinstein book. In the exercises followed by a *, restrict attention to pure strategies.
   
   (a) (Rationalizability=IESDS) 56.5.
   (b) (Pure Strategy NE) 19.1*. Assume that citizens are uniformly distributed, i.e. \( f(x) = 1 \).
   (c) (Mixed Strategy NE) 36.2. Assume that \( v_1 = 3, v_2 = 2, \) and \( v_3 = 1 \).
   (d) (Pure Strategy BNE) 28.1*.
   (e) (SPE) 103.2*.
   (f) (Sequential Equilibrium) 226.1.

3. Consider the signaling game in the last page.
   
   (a) Find a separating sequential equilibrium.
   (b) Find a pooling sequential equilibrium. Do not forget to specify beliefs at off the equilibrium path information sets. Is there a pooling equilibrium that survives the Intuitive Criterion?
   (c) Find a sequential equilibrium where a type of player 1 plays both L and R with positive probability.

Medium difficulty:

1. (Existence of Symmetric NE) A two-player finite normal form game \( G = (\{1, 2\}, A, u) \) is symmetric if (i) \( A_1 = A_2 \) and (ii) \( u_1(a, b) = u_2(b, a) \) for any \( a, b \in A_1 = A_2 \). A mixed strategy NE \( (\sigma_1, \sigma_2) \) of this game is symmetric if \( \sigma_1 = \sigma_2 \). Show that every symmetric two-player finite normal form game has a symmetric NE.

2. (Perfect Information Bargaining) Consider a modification of the alternating offers bargaining game studied in class where Player 1 makes offers in periods \( 3k + 1 \) and \( 3k + 2 \) and Player 2 makes offers in periods \( 3k + 3 \), for \( k = 0, 1, 2, \ldots \).
(a) Conjecture a SPE strategy profile of the players. (Hint: Suppose that the game has a unique SPE and let $V_i(l)$ be Player $i$’s SPE payoff in the subgame starting in period $3k+l$, where $i = 1, 2, l = 1, 2, 3,$ and $k = 0, 1, 2, \ldots$ One can express $(V_1(1), V_2(1))$ in terms of $(V_1(2), V_2(2))$, $(V_1(2), V_2(2))$ in terms of $(V_1(3), V_2(3))$, and $(V_1(3), V_2(3))$ in terms of $(V_1(1), V_2(1))$, solve for these six values, and identify the corresponding strategies.) Verify that the strategies you have found indeed form an SPE by using the single deviation property.

(b) Is there immediate agreement in equilibrium? Show that it is an advantage for player 1 to be able to make more frequent offers than in the original alternating offers model. Note in particular that as $\delta \to 1$, the cake is divided in proportion to number of offers made by each player.

(c) Show that the strategy profile you constructed in part (a) is the unique SPE.

3. Perfect Information Bargaining Consider a discrete version of the two player infinite horizon bargaining game studied in class where the set of feasible offers is a nonempty finite subset

$$A \subset \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0, x + y = 1\}.$$ 

Except for the change that $A$ is now finite, the rest of the game remains the same.\(^1\) Find $\bar{\delta} \in (0, 1)$ such that for any $\delta_1, \delta_2 \in (\bar{\delta}, 1)$, and $(x^*, y^*) \in A$, there is a SPE where player 1 offers the division $(x^*, y^*)$ in period 1 and player 2 immediately accepts. How do you contrast this with the uniqueness result from the class?

4. Perfect Information Bargaining Consider an infinite horizon bargaining game with three players $N = \{1, 2, 3\}$. In each period $t$, one of the players is randomly selected to make an offer: Player 1 is selected with probability $\frac{1}{2}$, each one of Players 2 and 3 is selected with probability $\frac{1}{4}$. The selected Player $i$ offers a division of the cake $(x_t, y_t, z_t)$ where $x_t, y_t, z_t \geq 0$ and $x_t + y_t + z_t = 1$ ($x_t$ denotes player 1’s share $y_t$ denotes player 2’s share and $z_t$ denotes player 3’s share). The two other Players $j$ and $k$ observe $i$’s offer $(x_t, y_t, z_t)$, then $j$ and $k$ simultaneously accept or reject this offer. If both $j$ and $k$ accept then the division is carried out, if at least one of them rejects then the offer is rejected and they proceed to period $t + 1$.

\(^1\)That is, the payoffs, the timeline, and the disagreement payoff of $(0, 0)$ remains the same.
Players maximize discounted expected payoffs and have the common discount factor $\delta \in (0,1)$. If no offer is ever accepted, then each player receives a payoff of zero. The selection of who makes an offer is i.i.d. across periods.

(a) Conjecture an SPE where (as usual) players accept any division where they receive at least $\delta$ times their continuation payoff. Write down formally the strategy profile and verify that it is indeed an SPE by using the single deviation property.

(b) For any division $(x, y, z)$, construct an SPE where the cake is divided according to $(x, y, z)$ in the first period, no matter who makes the offer.

Difficult:

1. (Reputation) Consider the following two player extensive form game with perfect information. The game starts by player 1’s (the principal) decision to hire (H) or to not hire (N) player 2 (the agent). If player 1 does not hire, then the game ends with zero payoffs to both parties. If she chooses to hire, then player 2 decides to work (W) or to shirk (S) where:

   $$u_1(H, W) = 1, u_1(H, S) = -1, u_2(H, W) = 1, u_2(H, S) = 2.$$ 

Consider a finite repetition of the above game, where a single long-lived agent faces a sequence $1, \ldots, K$ of short-run principals. All players observe past history of actions, payoff of the $k$th principal is her payoff at stage $k$, payoff of the agent is the sum of his payoffs in all stages.

(a) What is the subgame perfect equilibrium of the repeated game?

(b) Suppose now that before the game starts, nature determines a type for the agent in $\{R(egular), D(iligent)\}$: $\text{Prob}(R) = 1 - \epsilon, \text{Prob}(D) = \epsilon > 0$. The diligent type agent always works if hired.$^2$ Payoffs and strategies available to the regular agent and the principals are same as before. The agent knows his type but the entrants don’t. Let $\mu_k$ denote the belief of the principals that the agent is the diligent type.

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$^2$You may assume that it is physically impossible for the diligent type to shirk.
i. Find a sequential equilibrium where the belief process satisfies:

\[
\mu_{K-l+1} = \begin{cases} 
\mu_{K-l} & \text{if the agent is not hired at } K-l \\
0 & \text{if } (H,S) \text{ at } K-l \\
\max \left\{ \left( \frac{1}{2} \right)^l, \mu_{K-l} \right\} & \text{if } (H,W) \text{ at } K-l 
\end{cases}
\]

for \( l = 1, \ldots, K - 1 \), given the initial condition \( \mu_1 = \epsilon \).

ii. What is the sequential equilibrium payoff of the regular agent?

iii. (Optional) Show that if \( \epsilon \neq \left( \frac{1}{2} \right)^l \) for \( l = 1, \ldots, K \), then on the equilibrium path strategies and beliefs in any sequential equilibrium is unique.

2. (Weak Domination Theorem) In a normal form game, an action \( a_i \) is weakly dominated to a mixed strategy \( \sigma_i \) if:

- \( \forall a_{-i} \in A_{-i} : u_i(\sigma_i, a_{-i}) \geq u_i(a_i, a_{-i}), \) and
- \( \exists a_{-i} \in A_{-i} : u_i(\sigma_i, a_{-i}) > u_i(a_i, a_{-i}) \), and

Prove that in a finite normal form game, an action \( a_i^* \) is never a best reply to any (possibly correlated) completely mixed conjecture \( \sigma_{-i} \) of \( i \), if and only if \( a_i^* \) is weakly dominated to a mixed strategy \( \sigma_i \).\(^3\)

\(^3\)The correlated conjecture \( \sigma_{-i} \in \Delta(A_{-i}) \) is completely mixed if \( \sigma_{-i}(a_{-i}) > 0 \) for all \( a_{-i} \in A_{-i} \).
A Signaling Game