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Microeconomics II

Haluk Ergin & David Levine

Dominant Strategy Mechanisms

A General Model

$N = \{1, \ldots, n\}$ agents.

Y: A set of (social) outcomes.

Θ_i: The set of types of agent i. ($\Theta = \Theta_1 \times \ldots \times \Theta_n$.)

The type of an agent is her private information.

The only role of a type $\theta_i \in \Theta_i$ is in determining the agent's preferences over Y, which WLOG we assume are represented by a utility function.

$U_i : Y \times \Theta_i \to \mathbb{R}$ is the utility function of agent i.

Hence $U_i(\cdot; \theta_i)$ represents agent i's preferences over Y when her type is $\theta_i \in \Theta_i$.

A **social choice function (SCF)** is a function $g : \Theta \to Y$.

Two General Questions

Suppose we commit to a SCF g, and naively ask agents their types. Do they have incentives to tell us the truth?

More generally, can we construct a game whose outcomes are in Y, such that when it is played among the agents, for every $\theta \in \Theta$ there is an equilibrium with outcome $g(\theta)$? (i.e. the game somewhat indirectly implements g.)

A **mechanism** is a pair (A, h) where:

- A_i is an action/report set for agent i and $A = A_1 \times \ldots \times A_n$.
- $h : A \to Y$ is an outcome function, which determines the social outcome as a function of the agents' reports.

(Θ, g) is a **direct (revelation)** mechanism.

The Equilibrium Concept

A profile $\mathbf{a}^* = (\mathbf{a}_i^*)_{i \in N}$ where $\mathbf{a}_i^* \colon \Theta_i \to A_i$ is a **dominant strategy equilibrium of the mechanism** (A, h) if for all $i \in N$, $\theta_i, \in \Theta_i$, $a_i \in A_i$, $a_{-i} \in A_{-i}$:

$$U_i(h(\mathbf{a}_i^*(\theta_i), a_{-i}); \theta_i) \geq U_i(h(a_i, a_{-i}); \theta_i).$$

The mechanism (A, h) **implements the social choice function** g **in dominant strategies** if it has a dominant strategy equilibrium $\mathbf{a}^* = (\mathbf{a}_i^*)_{i \in N}$ such that for all $\theta \in \Theta$:

$$g(\theta) = h(\mathbf{a}_1^*(\theta_1), \ldots, \mathbf{a}_n^*(\theta_n)).$$

For the direct mechanism (Θ, g), we say that a dominant strategy equilibrium $\mathbf{a}^* = (\mathbf{a}_i^*)_{i \in N}$ is in **truthful strategies** if $\mathbf{a}_i^*(\theta_i) = \theta_i$ for all $i \in N$, $\theta_i \in \Theta_i$.

The Revelation Principle for Dominant Strategy Mechanisms

Assume that the mechanism (A, h) *implements the social choice function* g *in dominant strategies. Then, the direct mechanism* (Θ, g) *truthfully implements* g *in dominant strategies.*

Useful Consequence: There is no loss of generality from restricting attention to direct mechanisms, and to truthful dominant strategy equilibria.

A Discouraging Result

Assumption (Richness of Types) For all $i \in N$ and any strict ranking $\succ$ of the outcomes in Y, there is a type θ_i such that $U_i(\cdot; \theta_i)$ ranks outcomes in Y according to $\succ$.

Theorem *(Gibbard-Sattherwaite) Assume Richness of Types and* $|g(\Theta)| \geq 3$. *Then, the direct mechanism* (Θ, g) *does not truthfully implement* g *in dominant strategies.*

Note: By the revelation principle, under the conditions of the Theorem, there is *no mechanism* that implements g in dominant strategies.

Can we obtain positive results?

YES! But we need to consider models with more structure on Y such that the richness assumption fails naturally:

- Assume Y is ordered and that the preferences of the agents are single-peaked with respect to that order.

- Assume $Y = X \times \mathbb{R}^n$ where $x \in X$ is a social decision, $t = (t_1, \ldots, t_n)$ denotes monetary transfers to agents. Assume utility functions are quasi-linear in transfers.

- Assume that Y denotes possible allocations of n distinct indivisible objects to the agents so that each receives exactly one object. Assume that agents are indifferent between allocations where they receive the same object.

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The Quasi-Linear Model

$Y = X \times \mathbb{R}^n$, where X is a set of social allocations/decisions.

$t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ denotes a vector of transfers, t_i denotes the transfer *to* agent i.

In this set-up, a SCF is a pair $g = (f, t)$ where

- $f : \Theta \rightarrow X$ is a **social decision function / allocation rule**.
- $t : \Theta \rightarrow \mathbb{R}^n$ is a **transfer scheme**.

We assume that the utility functions of the agents are **quasi-linear** in transfers, i.e. they are of the form:

$$U_i(x, t; \theta_i) = u_i(x, \theta_i) + t_i$$

for all $i \in N$, $\theta_i \in \Theta_i$, $x \in X$, and $t \in \mathbb{R}^n$.

Characterization of Ex-Post Efficiency

The allocation rule $f : \Theta \to X$ is **ex-post efficient** if there does not exist a type profile $\theta \in \Theta$, an allocation $x \in X$, and a transfer vector $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ such that:

(i) $\sum_{i=1}^{n} t_i = 0$, and

(ii) $u_i(x, \theta_i) + t_i > u_i(f(\theta), \theta_i)$ for any $i \in N$.

Proposition *An allocation rule f is ex-post efficient if and only if for any $\theta \in \Theta$ and $x \in X$:*

$$\sum_{i=1}^{n} u_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} u_i(x, \theta_i).$$

Dominant Strategy Implementation

For a given allocation rule $f : \Theta \to X$*, can we design a supplementary transfer scheme* $t\colon \Theta \to \mathbb{R}^n$ *such that the direct mechanism* $(\Theta; f, t)$ *truthfully implements* (f, t) *in dominant strategies?*

The answer will be: YES, IF f is ex-post efficient.

Notes:

-We know from the revelation principle that there is no loss of generality from restricting ourselves to direct mechanisms and truthful implementation.

-The direct mechanism $(\Theta; f, t)$ truthfully implements (f, t) in dominant strategies, if for any $\theta \in \Theta$, $i \in N$ and $\widehat{\theta}_i \in \Theta_i$:

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq u_i(f(\widehat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\widehat{\theta}_i, \theta_{-i}).$$

The Vickrey-Groves-Clark (VGC) Mechanism

Proposition *Let f be an ex-post efficient allocation rule, and for each $i \in N$ let $h_i \colon \Theta_{-i} \to \mathbb{R}$ be an arbitrary function. Define transfer scheme t by:*

$$t_i(\theta) = \sum_{j \neq i} u_j(f(\theta), \theta_j) - h_i(\theta_{-i}).$$

Then $(\Theta; f, t)$ truthfully implements (f, t) in dominant strategies.

The Pivotal Mechanism

This is the special case of the VGC mechanism where the transfer schemes are given by:

$$h_i(\theta_{-i}) = \max_{x \in X} \left[\sum_{j \neq i} u_j(x, \theta_j) \right]$$

$$t_i(\theta) = \sum_{j \neq i} u_j(f(\theta), \theta_j) - \max_{x \in X} \left[\sum_{j \neq i} u_j(x, \theta_j) \right]$$

Note that:

-The pivotal transfers are never strictly positive, implying that the social planner never has to contribute out of his pocket.

-An agent does not make a payment unless he is pivotal in the sense that his presence tips over the social outcome from the maximizer of $\sum_{j \neq i} u_j(x, \theta_j)$ to $f(\theta)$.

An Example: Allocation of an indivisible good

A single object is to be allocated to exactly one agent.

$\Theta_i = [0,1]$: agent i's possible valuations for the object. Reservation utility of not getting the object is zero.

$X = N = \{1, \ldots, n\}$ determines who receives the object:

$$u_i(x, \theta_i) = \begin{cases} \theta_i & \text{if } i = x \\ 0 & \text{if } i \neq x. \end{cases}$$

Sum of utilities:

$$\sum_{i=1}^{n} u_i(x, \theta_i) = \theta_x, \quad x \in X.$$

The allocation rule f is ex-post efficient iff it allocates the object to an agent who has the highest valuation.

The Pivotal Mechanism transfers:

$$
\begin{aligned}
t_i(\theta) &= \sum_{j \neq i} u_j(f(\theta), \theta_j) - \max_{x \in X} \left[\sum_{j \neq i} u_j(x, \theta_j) \right] \\
&= \begin{cases} 0 - \theta^2 = -\theta^2 & \text{if } i = f(\theta) \\ \theta^1 - \theta^1 = 0 & \text{if } i \neq f(\theta) \end{cases}
\end{aligned}
$$

where θ^1, and θ^2 respectively denote the highest and second highest valuations in $\theta = (\theta_1, \ldots, \theta_n)$.

$\Rightarrow$ The highest valuation agent receives the object and pays the second highest valuation. The second-price auction!

Budget balancedness

The transfer schedule $t : \Theta \to \mathbb{R}$ is **budget-balanced** if it does not require any net transfers by or to the social planner, i.e. if $\sum_{i=1}^{n} t(\theta_i) = 0$ for any type profile $\theta \in \Theta$.

The pivotal mechanism transfers in the previous example is not budget balanced. *Can we find a budget balanced VGC mechanism?*

$$\sum_{i=1}^{n} t_i(\theta) = (n-1) \sum_{i=1}^{n} u_i(f(\theta), \theta_i) - \sum_{i=1}^{n} h_i(\theta_{-i})$$

In the previous example with $n = 2$, can we find h_1, h_2 s.t.

$$0 = t_1(\theta) + t_2(\theta) = \max\{\theta_1, \theta_2\} - h_1(\theta_2) - h_2(\theta_1)$$

for all $\theta = (\theta_1, \theta_2) \in [0, 1]^2$?

NO!

NOS and Individual Rationality

The transfer schedule $t : \Theta \to \mathbb{R}$ satisfies **No Outside Subsidies (NOS)** if $\sum_{i=1}^{n} t(\theta_i) \leq 0$ for all $\theta \in \Theta$.

It is easy to find VGC transfer schemes satisfying NOS, e.g. the pivotal mechanism. The challenge is in satisfying both NOS and Individual Rationality:

Suppose $\underline{u_i}(\theta_i)$ denotes the utility of i of type θ_i, from not participating in the mechanism. Example: Some $x^* \in X$ could be the fall back outcome and $\underline{u_i}(\theta_i) = u_i(x^*, \theta_i)$.

The mechanism (f, t) is **Individually Rational (IR)** if

$$u_i(f(\theta), \theta_i) + t_i(\theta) \geq \underline{u_i}(\theta_i) \quad \forall i \in N, \theta \in \Theta.$$

Clearly if the reservation utilities are high then IR and NOS incompatible. Question: *Can we find a VGC mechanism that satisfies both IR and NOS in "natural" environments?*

Bilateral Trade

Consider the earlier example with $n = 2$. Suppose agent 1 is the buyer and agent 2 is the seller of the object. The fall back option is no trade: $x^* = 2$. The reservation utilities:

$$\underline{u}_1(\theta_1) = u_1(x^*, \theta_1) = 0 \quad \text{and} \quad \underline{u}_2(\theta_2) = u_2(x^*, \theta_2) = \theta_2.$$

The IR constraints:

$$\max\{\theta_1, \theta_2\} - h_1(\theta_2) \geq 0 \qquad (1)$$

$$\max\{\theta_1, \theta_2\} - h_2(\theta_1) \geq \theta_2 \qquad (2)$$

The NOS constraint:

$$\max\{\theta_1, \theta_2\} - h_1(\theta_2) - h_2(\theta_1) \leq 0 \qquad (3)$$

There is no VGC transfers that satisfy all three constraints for all $\theta = (\theta_1, \theta_2) \in [0,1]^2$! (also indirectly implied by the Myerson and Satterthwaite Theorem)