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Dominance and Rationalizability

\( \sigma_i \) weakly (strongly) dominates \( \sigma'_i \) if

\[ u_i(\sigma_i, s_{-i}) \geq (> ) u_i(\sigma'_i, s_{-i}) \]  with at least one strict

**Prisoner’s Dilemma Game**

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

a unique dominant strategy equilibrium (D,L)

this is Paretodominated? Does it occur?
Public Goods Experiment

Players randomly matched in pairs

May donate or keep a token

The token has a fixed commonly known public value of 15

It has a randomly drawn private value uniform on 10-20

$V = \text{private gain/public gain}$

So if the private value is 20 and you donate you lose 5, the other player gets 15; $V = -1/3$

If the private value is 10 and you donate you get 5 the other player gets 15; $V = +1/3$

Data from Levine/Palfrey, experiments conducted with caltech undergraduates

Based on Palfrey and Prisbey
<table>
<thead>
<tr>
<th>V</th>
<th>donating a token</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>100%</td>
</tr>
<tr>
<td>0.2</td>
<td>92%</td>
</tr>
<tr>
<td>0.1</td>
<td>100%</td>
</tr>
<tr>
<td>0</td>
<td>83%</td>
</tr>
<tr>
<td>-0.1</td>
<td>55%</td>
</tr>
<tr>
<td>-0.2</td>
<td>13%</td>
</tr>
<tr>
<td>-0.3</td>
<td>20%</td>
</tr>
</tbody>
</table>
Nash Equilibrium

**Definition**

players can anticipate on another’s strategies

\[ \sigma \text{ is a } \textit{Nash equilibrium} \text{ profile if for each } i \in 1, \ldots, N \]

\[ u_i(\sigma) = \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}) \]
Mixed Strategies: The Kitty Genovese Problem

Description of the problem

Model:

$n$ people all identical

benefit if someone calls the police is $x$

cost of calling the police is 1

Assumption: $x > 1$

Look for symmetric mixed strategy equilibrium where $p$ is probability of each person calling the police
\( \) \( p \) is the symmetric equilibrium probability for each player to call the police

each player \( i \) must be indifferent between calling the police or not
if \( i \) calls the police, gets \( x - 1 \) for sure.
If \( i \) doesn’t, gets 0 with probability \( (1 - p)^{n-1} \), gets \( x \) with probability \( 1 - (1 - p)^{n-1} \)

so indifference when \( x - 1 = x \left( 1 - (1 - p)^{n-1} \right) \)

solve for \( p = 1 - (1/x)^{1/(n-1)} \)

probability police is called
\[ 1 - (1 - p)^n = 1 - \left( \frac{1}{x} \right)^{n-1} \]
probability police are called

$x=10$

$n$

2  5  8  11  14  17  20
Coordination Games

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

three equilibria (U,L) (D,R) (.5U,.5R)

**too many equilibria?? introspection possible??**

too many equilibria?? introspection possible?

the rush hour traffic game – introspection clearly impossible, yet we seem to observe Nash equilibrium

equilibrium through learning?
Coordination Experiments
Van Huyck, Battalio and Beil [1990]

Actions $A = \{1, 2, \ldots, c\}$

Utility $u(a_i, a_{-i}) = b_0 \min(a_j) - ba_i$ where $b_0 > b > 0$

Everyone doing $a'$ the same thing is always a Nash equilibrium

$a' = \bar{c}$ is efficient

the bigger is $a'$ the more efficient, but the “riskier”

a model of “riskier” some probability of one player playing $a' = 1$

story of the stag-hunt game
\( \bar{e} = 7 \), 14-16 players

Treatments:  
A \( b_0 = 2b \)  
B \( b = 0 \)

In final period treatment A:  
77 subjects playing \( a_i = 1 \)  
30 subjects playing something else  
Minimum was always 1

In final period treatment B:  
87 subjects playing \( a_i = 7 \)  
0 playing something else
with two players $a_i = 7$ was more common
### 1/2 Dominance

Coordination Game

<table>
<thead>
<tr>
<th></th>
<th>L ($p_1$)</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U ($p_2$)</td>
<td>2,2</td>
<td>-10,0</td>
</tr>
<tr>
<td>D</td>
<td>0,-10</td>
<td>1,1</td>
</tr>
</tbody>
</table>
risk dominance:
indifference between U,D

\[ 2p_2 - 10(1 - p_2) = (1 - p_2) \]
\[ 13p_2 = 11, \quad p_2 = 11/13 \]

if U,R opponent must play equilibrium w/ 11/13
if D,L opponent must play equilibrium w/ 2/13

\( \frac{1}{2} \) dominance: if each player puts weight of at least \( \frac{1}{2} \) on equilibrium strategy, then it is optimal for everyone to keep playing equilibrium
(same as risk dominance in 2x2 games)
Correlated Equilibrium

Chicken

<table>
<thead>
<tr>
<th></th>
<th>6,6</th>
<th>2,7</th>
</tr>
</thead>
<tbody>
<tr>
<td>7,2</td>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>

three Nash equilibria (2,7), (7,2) and mixed equilibrium w/ probabilities (2/3, 1/3) and payoffs (4 2/3, 4 2/3)
correlated strategy

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6,6</td>
<td>2,7</td>
</tr>
<tr>
<td>7,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

is a correlated equilibrium giving utility (5,5)

What is public randomization?
Approximate Equilibria and Near Equilibria

- **exact:** \( u_i(s_i | \sigma_{-i}) \geq u_i(s_i' | \sigma_{-i}) \)

- **approximate:** \( u_i(s_i | \sigma_{-i}) + \varepsilon \geq u_i(s_i' | \sigma_{-i}) \)

- Approximate equilibrium can be very different from exact equilibrium

Radner’s work on finite repeated PD

gang of four on reputation

upper and lower hemi-continuity
A small portion of the population playing "non-optimally" may significantly change the incentives for other players causing a large shift in equilibrium behavior.
Quantal Response Equilibrium
(McKelvey and Palfrey)

propensity to play a strategy

\[ p_i(s_i) = \exp(\lambda_i u_i(s_i, \sigma_{-i})) \]

\[ \sigma_i(s_i) = p_i(s_i) / \sum_{s_i'} p_i(s_i') \]

as \( \lambda_i \to \infty \) approaches best response

as \( \lambda_i \to 0 \) approaches uniform distribution
Smoothed Best Response Correspondence Example

\[ L (\sigma_2(L) = q) \quad R \]

\[
\begin{array}{c|c|c}
\sigma_1(U) = p & 1,1 & 0,0 \\
\hline
0,0 & 1,1 \\
\end{array}
\]

![Graph showing smoothed best response correspondence example](image-url)

- **q**: ordinate axis
- **p**: abscissa axis
- **1’s br**: One-sided best response
- **2’s br**: Two-sided best response
**Goeree and Holt: Matching Pennies**

Symmetric

<table>
<thead>
<tr>
<th></th>
<th>50% (48%)</th>
<th>50% (52%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% (48%)</td>
<td>80,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (52%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
<tr>
<td></td>
<td>12.5% (16%)</td>
<td>87.5% (84%)</td>
</tr>
<tr>
<td>----------------</td>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td>50% (96%)</td>
<td>320,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (4%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
<tr>
<td>(80%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50% (8%)</td>
<td>44,40</td>
<td>40,80</td>
</tr>
<tr>
<td>50% (92%)</td>
<td>40,80</td>
<td>80,40</td>
</tr>
</tbody>
</table>
Extensive Form Games

Definition of Extensive Form Game

a finite game tree $X$ with nodes $x \in X$

nodes are partially ordered and have a single root (minimal element)
terminal nodes are $z \in Z$ (maximal elements)
Example: a simple simultaneous move game
**Behavior Strategies**

a *pure strategy* is a map from information sets to feasible actions 
\( s_i(h_i) \in A(h_i) \)

a *behavior strategy* is a map from information sets to probability distributions over feasible actions 
\( \pi_i(h_i) \in P(A(h_i)) \)

*Nature’s move* is a behavior strategy for Nature and is a fixed part of the description of the game

We may now define \( u_i(\pi) \)

*normal form* are the payoffs \( u_i(s) \) derived from the game tree
(1,1) (2,2) (3,3) (4,4)
L  R  L  R
U  D  U  D

<table>
<thead>
<tr>
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<th>L</th>
<th>R</th>
</tr>
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<tbody>
<tr>
<td>U</td>
<td>1,1</td>
<td>2,2</td>
</tr>
<tr>
<td>D</td>
<td>3,3</td>
<td>4,4</td>
</tr>
</tbody>
</table>
Kuhn’s Theorem:

every mixed strategy gives rise to a unique behavior strategy

The converse is NOT true
1 plays .5 U
behavior:  2 plays .5L at U; .5L at R
mixed:  2 plays .5(LL),.5(RR)
        2 plays .25(LL),.25(RL),.25(LR),.25(RR)

however: if two mixed strategies give rise to the same behavior strategy, they are equivalent, that is they yield the same payoff vector for each opponents profile $u(\sigma_i, s_{-i}) = u(\sigma'_i, s_{-i})$
Trembling Hand Perfection

Selten Game

<table>
<thead>
<tr>
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<th>L</th>
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<tbody>
<tr>
<td>U</td>
<td>-1,-1</td>
<td>2,0</td>
</tr>
<tr>
<td>D</td>
<td>1,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
subgame perfect
equilibria:
UR is subgame perfect
D and .5 or more L is Nash but not subgame perfect
can also solve by weak dominance
or by trembling hand perfection
Example of Trembling Hand not Subgame Perfect

\[
\begin{array}{ccc}
A & D \\
\text{Lu=Ld} & 2,1 & 2,1 \\
\text{Ru} & 3,3 & 0,2 \\
\text{Fd} & 1,0 & 0,2 \\
\frac{1}{n} & \frac{1}{n} & \frac{(n-1)}{2} \\
\end{array}
\]

Here Ld,D is trembling hand perfect but not subgame perfect
definition of the agent normal form

each information set is treated as a different player, e.g. 1a, 1b if player 1 has two information sets; players 1a and 1b have the same payoffs as player 1

extensive form trembling hand perfection is trembling hand perfection in the agent normal form

what is sequentiality??
Robustness – The Selten Game

genericity in normal form

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>-1,-1</td>
<td>2**,0**</td>
</tr>
<tr>
<td>D</td>
<td>1**,1*(±ε)</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Self Confirming Equilibrium

$s_i \in S_i$ pure strategies for $i$; $\sigma_i \in \Sigma_i$ mixed

$H_i$ information sets for $i$

$\overline{H}(\sigma)$ reached with positive probability under $\sigma$

$\pi_i \in \Pi_i$ behavior strategies

$\hat{\pi}(h_i|\sigma_i)$ map from mixed to behavior strategies

$\rho(\pi), \rho(\sigma) \equiv \rho(\hat{\pi}(\sigma))$ distribution over terminal nodes
$\mu_i$, a probability measure on $\Pi_{-i}$

$u_i(s_i | \mu_i)$ preferences

$\Pi_{-i}(\sigma_{-i} | J) \equiv \{ \pi_{-i} | \pi_i(h_i) = \hat{\pi}(h_i | \sigma_i), \forall h_i \in H_{-i} \cap J \}$
Notions of Equilibrium

Nash equilibrium

a mixed profile $\sigma$ such that for each $s_i \in \text{supp}(\sigma_i)$ there exist beliefs $\mu_i$ such that

- $s_i$ maximizes $u_i(\cdot|\mu_i)$
- $\mu_i(\Pi_{-i}(\sigma_{-i}|H)) = 1$

Unitary Self-Confirming Equilibrium

- $\mu_i(\Pi_{-i}(\sigma_{-i}|H(\sigma))) = 1$

(=$\text{Nash with two players}$)
Fudenberg-Kreps Example

\[ \begin{align*}
A_1, A_2 & \text{ is self-confirming, but not Nash} \\
\text{any strategy for 3 makes it optimal for either 1 or 2 to play down} \\
\text{but in self-confirming, 1 can believe 3 plays R; 2 that he plays L}
\end{align*} \]
Heterogeneous Self-Confirming equilibrium

- \( \mu_i(\prod_{-i}(\sigma_{-i}|H(s_i,\sigma))) = 1 \)

Can summarize by means of “observation function”

\[
J(s_i,\sigma) = H, \bar{H}(\sigma), \bar{H}(s_i,\sigma)
\]
Remark: In games with perfect information, the set of heterogeneous self-confirming equilibrium payoffs (and the probability distributions over outcomes) are convex
Ultimatum Bargaining Results

\[ \begin{align*}
1 & \xrightarrow{x} 2 \\
& \xrightarrow{($10.00-x,x)} \\
& \xrightarrow{(0,0)}
\end{align*} \]
### Raw US Data for Ultimatum

<table>
<thead>
<tr>
<th>x</th>
<th>Offers</th>
<th>Rejection Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.00</td>
<td>1</td>
<td>100%</td>
</tr>
<tr>
<td>$3.25</td>
<td>2</td>
<td>50%</td>
</tr>
<tr>
<td>$4.00</td>
<td>7</td>
<td>14%</td>
</tr>
<tr>
<td>$4.25</td>
<td>1</td>
<td>0%</td>
</tr>
<tr>
<td>$4.50</td>
<td>2</td>
<td>100%</td>
</tr>
<tr>
<td>$4.75</td>
<td>1</td>
<td>0%</td>
</tr>
<tr>
<td>$5.00</td>
<td>13</td>
<td>0%</td>
</tr>
</tbody>
</table>

| 27      |

US $10.00 stake games, round 10
<table>
<thead>
<tr>
<th>Trials</th>
<th>Rnd</th>
<th>Cntry</th>
<th>Case</th>
<th>Expected Loss</th>
<th>Max Gain</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pl 1</td>
<td>Pl 2</td>
<td>Both</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>10</td>
<td>US</td>
<td>H</td>
<td>$0.00</td>
<td>$0.67</td>
<td>$0.34</td>
</tr>
<tr>
<td>27</td>
<td>10</td>
<td>US</td>
<td>U</td>
<td>$1.30</td>
<td>$0.67</td>
<td>$0.99</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>USx3</td>
<td>H</td>
<td>$0.00</td>
<td>$1.28</td>
<td>$0.64</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>USx3</td>
<td>U</td>
<td>$6.45</td>
<td>$1.28</td>
<td>$3.86</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>Yugo</td>
<td>H</td>
<td>$0.00</td>
<td>$0.99</td>
<td>$0.50</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>Yugo</td>
<td>U</td>
<td>$1.57</td>
<td>$0.99</td>
<td>$1.28</td>
</tr>
<tr>
<td>29</td>
<td>10</td>
<td>Jpn</td>
<td>H</td>
<td>$0.00</td>
<td>$0.53</td>
<td>$0.27</td>
</tr>
<tr>
<td>29</td>
<td>10</td>
<td>Jpn</td>
<td>U</td>
<td>$1.85</td>
<td>$0.53</td>
<td>$1.19</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>Isrl</td>
<td>H</td>
<td>$0.00</td>
<td>$0.38</td>
<td>$0.19</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>Isrl</td>
<td>U</td>
<td>$3.16</td>
<td>$0.38</td>
<td>$1.77</td>
</tr>
<tr>
<td>WC</td>
<td></td>
<td></td>
<td>H</td>
<td>$5.00</td>
<td></td>
<td>$10.00</td>
</tr>
</tbody>
</table>

Rnds=Rounds, WC=Worst Case, H=Heterogeneous, U=Unitary
Comments on Ultimatum

- every offer by player 1 is a best response to beliefs that all other offers will be rejected so player 1’s heterogeneous losses are always zero.
- big player 1 losses in the unitary case
- player 2 losses all knowing losses from rejected offers; magnitudes indicate that subgame perfection does quite badly
- as in centipede, tripling the stakes increases the size of losses a bit less than proportionally (losses roughly double).
This game has a unique self-confirming equilibrium; in it player 1 with probability 1 plays $T_1$.
Summary of Experimental Results
<table>
<thead>
<tr>
<th>Trials / Rnd</th>
<th>Rnds</th>
<th>Stake</th>
<th>Case</th>
<th>Expected Loss</th>
<th>Max Gain</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Pl 1</td>
<td>Pl 2</td>
<td>Both</td>
</tr>
<tr>
<td>29*</td>
<td>6-10</td>
<td>1x</td>
<td>H</td>
<td>$0.00</td>
<td>$0.03</td>
<td>$0.02</td>
</tr>
<tr>
<td>29*</td>
<td>6-10</td>
<td>1x</td>
<td>U</td>
<td>$0.26</td>
<td>$0.17</td>
<td>$0.22</td>
</tr>
<tr>
<td>WC</td>
<td>1x</td>
<td>H</td>
<td></td>
<td></td>
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<td>$0.80</td>
</tr>
<tr>
<td>29</td>
<td>1-10</td>
<td>1x</td>
<td>H</td>
<td>$0.00</td>
<td>$0.08</td>
<td>$0.04</td>
</tr>
<tr>
<td>10</td>
<td>1-10</td>
<td>4x</td>
<td>H</td>
<td>$0.00</td>
<td>$0.28</td>
<td>$0.14</td>
</tr>
</tbody>
</table>
Rnds=Rounds, WC=Worst Case, H=Heterogeneous, U=Unitary
*The data on which from which this case is computed is reported above.
Comments on Experimental Results

- heterogeneous loss per player is small; because payoffs are doubling in each stage, equilibrium is very sensitive to a small number of player 2’s giving money away at the end of the game.
  - unknowing losses far greater than knowing losses
  - quadrupling the stakes very nearly causes $\varepsilon$ to quadruple
  - theory has substantial predictive power: see WC
  - losses conditional on reaching the final stage are quite large--inconsistent with subgame perfection. McKelvey and Palfrey estimated an incomplete information model where some “types” of player 2 liked to pass in the final stage. This cannot explain many players dropping out early so their estimated model fits poorly.