

**Econ 504 (2008)
Microeconomics II**

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Auctions I

Auction Models

- **Single** versus multiple objects.
- Private versus common values.
- Private values: Independent versus correlated.
- Symmetric versus asymmetric.

Popular Auction rules

- Dynamic:
 - Ascending-price (or English) auction.
 - Descending-price (or Dutch) auction.
- Static:
 - First-price sealed-bid auction.
 - Second-price sealed-bid auction.

Equivalent auctions:

- Strong equivalence: Descending-price and the first-price sealed-bid are strategically equivalent.
- Weak equivalence: Optimal strategies in ascending-price and second-price sealed-bid auctions are equivalent for private values.

The Symmetric Independent Private Values Model

Aside: Order statistics

Let X_1, \dots, X_n be n random variables.

For $k = 1, \dots, n$, the **k th order statistic** is the random variable X^k that takes the k th largest value among X_1, \dots, X_n , counting repetitions of the same values.

The first order statistic is $X^1 = \max\{X_1, \dots, X_n\}$.

The second order statistic X^2 takes the second highest value in X_1, \dots, X_n .

Example If $n = 4$ and (X_1, X_2, X_3, X_4) take the value $(5, 6, 10, 6)$, then the order statistics (X^1, X^2, X^3, X^4) take the values $(10, 6, 6, 5)$.

The model

$N = \{1, 2, \dots, n\}$ bidders, bidding for a single indivisible object.

Bidder i 's valuation for object i is denoted by X_i or x_i .

The bidders' valuations are i.i.d. with density f and c.d.f. F , in a closed interval $[\underline{x}, \bar{x}] \subset \mathbb{R}$. f is continuous and $f(x) > 0$ for any $x \in [\underline{x}, \bar{x}]$. ($\Rightarrow F$ is strictly increasing)

Sealed bids: A strategy of bidder i is a bid function $b_i : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$, where $b_i(x_i)$ specifies how much bidder i bids for the object when his type/valuation is x_i . (pure strategies!)

Note: Independence, symmetry, private values, risk neutrality.

First-price sealed-bid auction rule

The highest bidder wins the object and pays his bid.

If there are multiple highest bidders, the object goes to each such bidder with equal probability.

The payoff function:

$$u_i(b_1, \dots, b_n; x_i) = \begin{cases} \frac{1}{|\{j \in N: b_j = b^1\}|} (x_i - b_i) & \text{if } b_i = b^1 \\ 0 & \text{otherwise.} \end{cases}$$

b^1 denotes the highest bid (the first-order statistic) among b_1, \dots, b_n .

Reservation utility when the object is not received and no payment is made is normalized to be zero.

Second-price sealed-bid auction rule

The highest bidder wins the object and pays the second highest bid.

If there are multiple highest bidders, the object goes to each such bidder with equal probability.

The payoff function:

$$u_i(b_1, \dots, b_n; x_i) = \begin{cases} \frac{1}{|\{j \in N: b_j = b^1\}|} (x_i - b^2) & \text{if } b_i = b^1 \\ 0 & \text{otherwise.} \end{cases}$$

b^2 denotes the second highest bid (the second-order statistic) among b_1, \dots, b_n .

Again, reservation utility=0.

Second-price auction: Dominant strategy equilibrium

Theorem *In the second-price auction, it a dominant strategy for a bidder i with valuation $x_i \in [\underline{x}, \bar{x}]$ to bid:*

$$b_i(x_i) = x_i.$$

That is for each b'_i :

$$u_i(x_i, b_{-i}; x_i) \geq u_i(b'_i, b_{-i}; x_i)$$

for all b_{-i} and the inequality is strict for some b_{-i} .

The dominant strategy equilibrium is unique but there are other Bayesian Nash equilibria!

Equilibrium bid function is strictly increasing and the cdf's have a density, hence ties have zero probability in the dominant strategy equilibrium.

Observations

In the dominant strategy equilibrium, for a bidder with valuation $x \in [\underline{x}, \bar{x}]$:

Probability of winning:

$$P(X_{-i}^1 < x) = P(X_{-i}^1 \leq x) = F^{n-1}(x)$$

where $X = (X_1, \dots, X_n)$.

Expected payment conditional on winning:

$$E[X_{-i}^1 | X_{-i}^1 \leq x].$$

First-price auction: Characterization of the Symmetric Bayesian-Nash equilibrium

Our strategy will be “conjecture and verify”. We will conjecture that there is a **symmetric** BNE, i.e. one where every bidder uses the same bid function $b_i(x_i) = b(x_i)$.

Three main steps of the proof:

1. Prove that $b(\cdot)$ is differentiable on $(\underline{x}, \bar{x}]$, and has strictly positive derivative.
2. Derive a differential equation satisfied by $b(\cdot)$ and solve for it “uniquely.”
3. Prove that the derived $b(\cdot)$ indeed induces a Bayesian Nash equilibrium.

Step 1: b is differentiable

For any $\bar{b} \in \mathbb{R}$, let $Q(\bar{b})$ denote the probability that i wins when he bids \bar{b} and the others bid according to $b(\cdot)$. Note Q is nondecreasing.

For any $x, y \in [\underline{x}, \bar{x}]$:

$$\begin{aligned}(x - b(x))Q(b(x)) &\geq (x - b(y))Q(b(y)) \\ (y - b(y))Q(b(y)) &\geq (y - b(x))Q(b(x))\end{aligned}$$

implying

$$(x - y)[Q(b(x)) - Q(b(y))] \geq 0.$$

$$(1.1) \quad x > y \Rightarrow Q(b(x)) \geq Q(b(y)).$$

$$(1.2) \quad b(x) > b(y) \ \& \ Q(b(x)) = Q(b(y)) \Rightarrow Q(b(x)) = 0.$$

$$(1.3) \quad b \text{ is nondecreasing.}$$

(1.4) b is strictly increasing. Therefore ties occur with zero probability and $Q(b(x)) = F^{n-1}(x)$.

(1.5) b is continuous on $(\underline{x}, \bar{x}]$ and $b(\underline{x}) \leq \inf_{\underline{x} < y \leq \bar{x}} b(y)$.

(1.6) b is differentiable on $(\underline{x}, \bar{x}]$ (only left differentiable at \bar{x}) and $b' > 0$.

Aside: Derivative of an inverse function

(a, b) , (c, d) open intervals in \mathbb{R} . $g: (a, b) \rightarrow (c, d)$ is onto, differentiable and $g' > 0$.

The inverse function, $g^{-1}: (c, d) \rightarrow (a, b)$ is defined by:

$$g^{-1}(y) = x \iff g(x) = y.$$

Then:

$$\frac{d}{dy}g^{-1}(y) = \frac{1}{g'(g^{-1}(y))}.$$

Step 2: Solve for b

Payoff of agent i with valuation x from bidding $b_i \in \mathbb{R}$:

$$F^{n-1}(b^{-1}(b_i))(x - b_i)$$

Necessary first order condition for maximization:

$$\begin{aligned} 0 &= \frac{d}{db_i} \left[F^{n-1}(b^{-1}(b_i))(x - b_i) \right] \\ &= (n-1)F^{n-2}(b^{-1}(b_i))f(b^{-1}(b_i))\frac{1}{b'(b^{-1}(b_i))}(x - b_i) \\ &\quad - F^{n-1}(b^{-1}(b_i)). \end{aligned}$$

Evaluate at $b_i = b(x)$, then $b^{-1}(b_i) = x$:

$$0 = (n-1)F^{n-2}(x)f(x)\frac{1}{b'(x)}(x - b(x)) - F^{n-1}(x).$$

Rewrite as:

$$\begin{aligned} & (n-1)F^{n-2}(x)f(x)x + F^{n-1}(x) - F^{n-1}(x) \\ &= b'(x)F^{n-1}(x) + (n-1)F^{n-2}(x)f(x)b(x), \end{aligned}$$

By the product rule:

$$\frac{d}{dx} [F^{n-1}(x)x] - F^{n-1}(x) = \frac{d}{dx} [b(x)F^{n-1}(x)].$$

Integrating from \underline{x} to x :

$$\begin{aligned} & F^{n-1}(x)x - F^{n-1}(\underline{x})\underline{x} - \int_{\underline{x}}^x F^{n-1}(\alpha)d\alpha \\ &= b(x)F^{n-1}(x) - b(\underline{x})F^{n-1}(\underline{x}). \end{aligned}$$

The solution:

$$b(x) = x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^x F^{n-1}(\alpha)d\alpha.$$

Step 3: Verify b is a BNE

It is never optimal to bid more than $b(\bar{x})$. Any bid strictly less than $b(\underline{x})$ gives an same (zero) payoff as bidding $b(\underline{x})$. By continuity $b([\underline{x}, \bar{x}]) = [b(\underline{x}), b(\bar{x})]$.

Payoff of bidder with valuation x from bidding $b(y)$:

$$\begin{aligned} [x - b(y)]F^{n-1}(y) &= [y - b(y)]F^{n-1}(y) + (x - y)F^{n-1}(y) \\ &= \int_{\underline{x}}^y F^{n-1}(\alpha) d\alpha + (x - y)F^{n-1}(y) \\ &= \int_{\underline{x}}^y F^{n-1}(\alpha) d\alpha + \int_y^x F^{n-1}(y) d\alpha \\ &= \int_{\underline{x}}^x F^{n-1}(\alpha) d\alpha + \int_y^x [F^{n-1}(y) - F^{n-1}(\alpha)] d\alpha \\ &= [x - b(x)]F^{n-1}(x) + \int_y^x [F^{n-1}(y) - F^{n-1}(\alpha)] d\alpha \end{aligned}$$

Since F is strictly increasing, the final integral is negative if $x < y$ or if $y < x$.

We proved...

Theorem *In the first price auction, there is a “unique” symmetric Bayesian Nash equilibrium given by the bid function:*

$$b(x) = x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^x F^{n-1}(\alpha) d\alpha.$$

for all $x \in [\underline{x}, \bar{x}]$.

$b(\underline{x})$ is not pinned down any more than $b(\underline{x}) \leq \underline{x}$.

Bid shading disappears (i.e. $b(x) \rightarrow x$) as $n \rightarrow \infty$:

$$\int_{\underline{x}}^x \left(\frac{F(\alpha)}{F(x)} \right)^{n-1} d\alpha \rightarrow 0$$

since $\left(\frac{F(\alpha)}{F(x)} \right)^{n-1} \rightarrow 0$ for all $\alpha < x$.

Example: The uniform distribution

$[\underline{x}, \bar{x}] = [0, 1]$ and $f(x) = 1 \Rightarrow F(x) = x$.

First price BNE bid function:

$$b(x) = x - \frac{1}{x^{n-1}} \int_0^x \alpha^{n-1} d\alpha = \left(1 - \frac{1}{n}\right) x$$

Observation

Note $F_{X_{-i}^1}(x) = \mathbb{P}(X_{-i}^1 \leq x) = F^{n-1}(x)$. Then:

$$\begin{aligned} b(x) &= x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^x F^{n-1}(\alpha) d\alpha \\ &= \frac{1}{F^{n-1}(x)} \left[xF^{n-1}(x) - \underline{x}F^{n-1}(\underline{x}) - \int_{\underline{x}}^x F^{n-1}(\alpha) d\alpha \right]. \\ &= \frac{1}{F^{n-1}(x)} \left[\int_{\underline{x}}^x \alpha \left(\frac{d}{d\alpha} F^{n-1}(\alpha) \right) d\alpha \right]. \\ &= \frac{1}{F_{X_{-i}^1}(x)} \left[\int_{\underline{x}}^x \alpha f_{X_{-i}^1}(\alpha) d\alpha \right]. \\ &= \mathbb{E}[X_{-i}^1 | X_{-i}^1 \leq x] \end{aligned}$$

The bidder i with valuation x bids the expected value of the highest valuation of the other bidders (equivalently of the overall second highest value) conditional on i winning.

Payoff and Revenue comparisons

Let $m^k(x)$ (for $k = I, II$) be the **expected payment** of a bidder with valuation x in the k th price auction equilibrium. In general expected payment of x is:

$$\mathbb{P}(x \text{ wins}) \times \mathbb{E}[\text{payment of } x \mid x \text{ wins}].$$

Therefore:

$$m^I(x) = F^{n-1}(x) \times \mathbb{E}[X_{-i}^1 \mid X_{-i}^1 \leq x] = m^{II}(x).$$

Implication 1: Payoff equivalence

Payoff of a bidder with valuation x in the k th price auction:

$$\mathbb{P}(x \text{ wins in the } k\text{th price auction eqm}) \times x - m^k(x).$$

Since $\mathbb{P}(x \text{ wins}) = F^{n-1}(x)$ in both auction equilibria and $m^I(x) = m^{II}(x)$, *payoff of a bidder with valuation x is the same in both auctions.*

Implication 2: Revenue equivalence

The **ex-ante expected payment** of a bidder (i.e. before her valuation is drawn) in the k th price auction:

$$m^k = \mathbb{E}[m^k(X_i)] = \int_{\underline{x}}^{\bar{x}} m^k(x) f(x) dx$$

is the same in both auctions since $m^I(x) = m^{II}(x)$ for all x . Therefore *the expected revenue of the seller $n \times m^k$ is the same in the two auctions.*