Econ 504 Problem 3
Suggested Answers

Static Games
(Throughout these questions we denote the row player by player 1 with strategies $U$, $M$, $D$ and the column player by player 2 with strategies $l$, $m$, $r$ with respect to their positions.)

a) There are no dominated strategies. Two pure Nash equilibria are $(U, l)$ and $(D, r)$, which are trembling hand perfect because they are also strict Nash Equilibria. There exists another mixed Nash equilibrium $(\frac{2}{3}U + \frac{1}{3}D, \frac{1}{2}l + \frac{2}{3}r)$, which is also trembling hand perfect because it is completely mixed.

b) There is a strict dominant strategy equilibrium $(D, r)$, which is the unique Nash equilbrium. It is trembling hand perfect because it is strict.

c) Notice that $m$ is strictly dominated by $l$ and hence can be eliminated. Given that $m$ is never played, both $M$ and $D$ are dominated by $M$. But then $r$ can be deleted as well. Thus $(U, l)$ is the only strategy profile that survives iterated strict dominance and the hence only Nash equilibrium. It is trembling hand perfect since it is strict.

d) There are no dominated strategies. Two pure Nash equilibria are $(U, l)$ and $(D, r)$. There are both trembling hand perfect because $U$ is a strict best reply to $l$, $D$ is a strict best reply to $r$, and both $l$ and $r$ are best replies to any mixed strategies of player 1. Hence they are still best replies to a small perturbation of the opponent’s strategies. In addition, $(pU + (1-p)D, \frac{1}{2}l + \frac{1}{2}r)$, $p \in [0,1]$ are also Nash equilibria. Again they are all trembling hand perfect: $\sigma_1$ is a best reply to a completely mixed strategy, and $\sigma_2$ is a best reply to any mixed strategies of player 1.

Dynamic Games
First denote the feasible actions at each information set as in Figure 1. Note
that player 1 has 8 pure strategies, namely \{agk, agl, ahk, ahl, bgk, bgl, bhk, bhl\}, and player 2 also has 8 pure strategies, i.e., \{cei, cej, cfi, cfj, dei, dej, dfi, dfj\}.

1) The normal form of the game can be summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>cei</th>
<th>cej</th>
<th>cfi</th>
<th>cfj</th>
<th>dei</th>
<th>dej</th>
<th>dfi</th>
<th>dfj</th>
</tr>
</thead>
<tbody>
<tr>
<td>agk</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>agl</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>ahk</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>ahl</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>bgk</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
</tr>
<tr>
<td>bgl</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
</tr>
<tr>
<td>bhk</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>1,5</td>
<td>2,2</td>
<td>1,5</td>
<td>2,2</td>
</tr>
<tr>
<td>bhl</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>7,3</td>
<td>0,5</td>
<td>2,2</td>
<td>0,5</td>
<td>2,2</td>
</tr>
</tbody>
</table>

2) Notice that \{agk, agl, ahk, ahl\} are strategic equivalent, i.e., they gen-
erate the same payoffs for both players in all situations, and so does any mixed strategies among them. Therefore we can represent them just by a generic strategy $a$, which means any probability distribution over these 4 strategies. Similarly we can denote \{bgk, bgl\} by bg, \{cei, cej\} by ce and \{cfi, cfj\} by cf and write down the reduced normal form:

<table>
<thead>
<tr>
<th></th>
<th>ce</th>
<th>cf</th>
<th>dei</th>
<th>dej</th>
<th>dfi</th>
<th>dfj</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>5,5</td>
<td>0,0</td>
<td>5,5</td>
<td>5,5</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>bg</td>
<td>7,3</td>
<td>7,3</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
</tr>
<tr>
<td>bhk</td>
<td>7,3</td>
<td>7,3</td>
<td>1,5,4</td>
<td>2,2</td>
<td>1,5,4</td>
<td>2,2</td>
</tr>
<tr>
<td>bhl</td>
<td>7,3</td>
<td>7,3</td>
<td>0,5,1</td>
<td>2,2</td>
<td>0,5,1</td>
<td>2,2</td>
</tr>
</tbody>
</table>

The pure Nash equilibria of the reduced form game are $(a, dei)$, $(a, dej)$, $(bg, ce)$, $(bg, cf)$ $(bhk, ce)$, $(bhl, cf)$, and $(bhk, dfi)$.

Notice that any convex combination between $(a, dei)$, $(a, dej)$ is still a (mixed) Nash equilibrium, and so are convex combinations between $(bg, ce)$, $(bg, cf)$ and convex combinations between $(bhk, ce)$ and $(bhl, cf)$. So we can denote them more compactly as $(a, de)$, $(bg, c)$, and $(bhl, c)$ with the understanding that whenever a generic strategy is used, it corresponds to any convex combination of that family of strategies.

Are there other mixed Nash equilibria? First, notice that we can eliminate $dfj$ because it is strict dominated by $ce$. Secondly, we claim that there is no mixed equilibria such that $0 < \sigma_1(a) < 1$. We argue it by contradiction.

Suppose that there is an equilibrium $(\sigma_1, \sigma_2)$ such that $0 < \sigma_1(a) < 1$. Then $cf$ and $dej$ are strictly worse than $ce$, and $dfi$ is strictly worse than $dei$, so they cannot be played in equilibrium. Hence it must be $\sigma_2(ce) + \sigma_2(dei) = 1$. For $\sigma_2(ce) = 1$, $a$ is strictly worse than other strategies, and for $\sigma_2(ce) = 0$, $a$ is strictly better than other strategies. Since $0 < \sigma_1(a) < 1$, both cases cannot hold. But if $0 < \sigma_2(ce) < 1$, then $bg$ and $bhl$ are strictly worse than $bhk$, so again they cannot be played in equilibrium. Hence it must be $\sigma_1(a) + \sigma_1(bhk) = 1$. But than by the same argument $dei$ is strictly worse
than $ce$, so $dei$ cannot be played in equilibrium, contradict to the previous result that $0 < \sigma_2(ce) < 1$.

We already know that the set of equilibria such that $\sigma_1(a) = 1$ can be represented by $(a, de)$, so let’s focus on the case $\sigma_1(a) = 0$. Notice that if $a$ is not played, then $dej$ is strictly worse than $ce$. Thus we can focus on the submatrix:

$$
\begin{array}{cccc}
& ce & cf & dei & dfi \\
bg & 7,3 & 7,3 & 1,2 & 1,2 \\
bhk & 7,3 & 7,3 & 1.5,4 & 1.5,4 \\
bhl & 7,3 & 7,3 & 0.5,1 & 0.5,1 \\
\end{array}
$$

Notice that in this submatrix $ce$ is equivalent to $cf$ and $dei$ is equivalent to $dfi$, so we denote them as $c$ and $di$ respectively. Notice that if $\sigma_2(di) > 0$, then $bg$ and $bhl$ are strictly worse than $bhk$, and hence in equilibrium $\sigma_1(bhk) = 1$. Player 2’s best replies are $dei$ and $dfi$, and he can mix between them as long as $bhk$ is still better than $a$ for player 1. That gives the condition $\sigma_2(dei) \leq 0.3$.

For $\sigma_2(di) = 0$, player 1 is indifferent between $bg$, $bhk$, and $bhl$. Thus, player 1 can mix between them as long as $di$ is no better than $c$ for player 2. That gives the condition $\sigma_1(bg) + 3\sigma_1(bhk) \leq 2$. Player 2 can mix arbitrarily between $ce$ and $cf$ without causing player 1 to choose $a$.

In sum, we can represent the set of Nash equilibria by $(a, de)$, $(bhk, \sigma_2)$ such that $\sigma_2(di) = 1$, $\sigma(dei) \leq 0.3$, and $(\sigma_1, c)$ such that $\sigma_1(a) = 0$, $\sigma_1(bg) + 3\sigma_1(bhk) \leq 2$.

3) For the subgame starting with the move by nature, there are two possible equilibrium payoffs: $(1.5, 4)$ and $(2, 2)$, which correspond to the strategy profiles $(k,i)$, $(Pr(l) \geq 2/3, j)$, respectively. By using backward induction, the former payoff gives a subgame perfect equilibrium $(\sigma_1 = (a, h, k), \sigma_2 = (d, e, i))$ with payoff $(5, 5)$. Similarly the later payoff gives another set of subgame perfect equilibria $(\sigma_1 = (b, h, Pr(l) \geq 2/3), \sigma_2 = (c, e, j))$ with payoff
Dominance and Nash Equilibrium

(Throught the question I will keep the assumption that the game is finite.)

Let’s begin with an example that iterated removal of weak dominated strategies could rule out some Nash equilibria. Consider the following normal form game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2</td>
<td>2,2</td>
</tr>
<tr>
<td>D</td>
<td>3,1</td>
<td>1,0</td>
</tr>
</tbody>
</table>

There are two pure Nash equilibrium: \((U, r)\) and \((D, l)\). But notice that \(r\) is weakly dominated by \(l\), and if \(r\) has been removed then \(U\) is dominated by \(D\), leaving only \((D, l)\).

One observation to this example is that deletion of weakly dominated strategies may rule out a strategy that has been played with positive probability in equilibrium, and that cannot happen if we still want the strategy to be an equilibrium in the reduced game. Indeed, it is part of the proof in which iterated strict dominance is used.

(Iterated strict dominance): Assume now that \(\sigma\) is the Nash equilibrium of a game, and we claim that it is also a Nash equilibrium of the game in

1\(^{\text{With infinite actions some funny things may happen, for instance a Nash equilibrium of the reduced game may not be a Nash equilibrium of the original game. Consider the following example:}}\)

<table>
<thead>
<tr>
<th></th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(\ldots)</th>
<th>(a_n)</th>
<th>(\ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1, 1.5</td>
<td>1, 0</td>
<td>1, 1/2</td>
<td>1, 2/3</td>
<td>(\ldots)</td>
<td>1, (1 - 1/n)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>D</td>
<td>2, 1.5</td>
<td>2, 1</td>
<td>2, 3/2</td>
<td>2, 5/3</td>
<td>(\ldots)</td>
<td>2, (2 - 1/n)</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

See that \(a_k\) is strictly dominated by \(a_{k+1}\) for all \(k \geq 1\), so the only undominated strategy is \(a_0\). Then \(U\) is dominated by \(D\), leaving only one strategy profile \((D, a_0)\). But \(a_0\) is not a best reply to \(D\) in the original game! There are some other abnormal features of this example: the original game has no Nash equilibrium, and the sequence of elimination matters. See that \(U\) is also strictly dominated by \(D\), and after removing \(U\) every \(a_k\) is a strictly dominated strategy. Can you fix the game to make it behave nicely?
which strategies have been removed by iterated strict dominance. Denote $S^0$ to be the set of strategy profiles of the original game and $S^t$ to be the set of strategy profiles that survive the $t$-th elimination of iterated strictly dominated strategies. Since the game is finite, there exists some $T$ such that $S^t = S^{t'} = S^\infty$ for all $t, t'$ greater than $T$, and the set $S^\infty$ is the set of pure strategies that survive iterated strict dominance. Now, if we can show that $\sigma$ is a Nash equilibrium for the $t$-th game implies that it is also a Nash equilibrium for the $(t + 1)$-th game, then repeating this argument for $t = 0, \ldots, T$ we get that $\sigma$ is a Nash equilibrium of the $(T + 1)$-th game if it is a Nash equilibrium of the original game, and hence the equilibrium of the game with strategies that survive iterated strict dominance.

So let’s assume that $\sigma$ is a Nash equilibrium of the $t$-th game. Therefore,

$$\forall i \quad (\forall s_i \text{ s.t. } \sigma_i(s_i) > 0) \quad u_i(s_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \quad \forall \sigma'_i \in \Delta(S^t_i).$$

To show that $\sigma$ is a Nash equilibrium of the $(t + 1)$-th game, first we need to show that $\sigma_i \in \Delta(S^{t+1}_i)$ for all $i$, otherwise $\sigma$ is even not a strategy profile for the $(t + 1)$-th game. Now if $S^{t+1}_i = S^t_i$ then there is nothing to prove. If $S^{t+1}_i \neq S^t_i$, then for any $s_i$ in $S^t_i$ but not belongs to $S^{t+1}_i$ there is a mixed strategy $\hat{\sigma}_i \in \Delta(S^t_i)$ that strictly dominates $s_i$ in the $t$-th game. Thus $u_i(\hat{\sigma}_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$. But that means $s_i$ is not a best reply to $\sigma_{-i}$, which implies $\sigma_i(s_i) = 0$. Hence, $\sigma_i \in \Delta(S^{t+1}_i)$ for all $i$.

What remains to be shown is that $\sigma_i$ is a best reply to $\sigma_{-i}$ in the $(t + 1)$-th game. That is,

$$\forall i \quad (\forall s_i \text{ s.t. } \sigma_i(s_i) > 0) \quad u_i(s_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \quad \forall \sigma'_i \in \Delta(S^{t+1}_i),$$

which is obviously true since $S^{t+1}_i \subseteq S^t_i$.

Now we claim that the converse is also true: $\sigma$ is the Nash equilibrium of a game if it is a Nash equilibrium of the game in which strategies have been removed by iterated strict dominance. To this end we need to prove a lemma similar to the above one but in the reverse order: if $\sigma$ is a Nash equilibrium for the $t$-th game then that it is also a Nash equilibrium for the $(t - 1)$-th
game. If it is true, then start with the $T$-th game and repeat this argument to $t = T, T-1, \cdots, 1$ we get the desired result.

Now we prove the lemma. Let $\sigma$ be a Nash equilibrium for the $t$-th game. Then

$$(\forall i) \ u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}) \ \forall s_i \in S^t_i.$$  

We need to show that

$$(\forall i) \ u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}) \ \forall s_i \in S^{t-1}_i.$$  

Pick any $i$, the above is obviously true if $S^{t-1}_i = S^t_i$. If $S^{t-1}_i \neq S^t_i$, then since the original game is finite, there exist $\hat{s}_i \in D^t_i \equiv S^{t-1}_i - S^t_i$ and $\hat{\sigma}_i \in \Delta(S^{t-1}_i)$ such that for all $s_i \in D^t_i$,

$$u_i(s_i, \sigma_{-i}) \leq u_i(\hat{s}_i, \sigma_{-i}) \leq u_i(\hat{\sigma}_i, \sigma_{-i}).$$  

From the second inequality we can write

$$u_i(\hat{s}_i, \sigma_{-i}) \leq \sum_{s_i \in D^t_i} \hat{\sigma}_i(s_i) u_i(s_i, \sigma_{-i}) + \sum_{s_i \in S^t_i} \hat{\sigma}_i(s_i) u_i(s_i, \sigma_{-i})$$

$$\leq \left[\sum_{s_i \in D^t_i} \hat{\sigma}_i(s_i)\right] u_i(\hat{s}_i, \sigma_{-i}) + \left[\sum_{s_i \in S^t_i} \hat{\sigma}_i(s_i)\right] u_i(\hat{\sigma}_i, \sigma_{-i})$$

$$\leq u_i(\sigma_i, \sigma_{-i}),$$

but that implies $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$ for all $s_i \in S^{t-1}_i$, so $\sigma_i$ is a best reply to $\sigma_{-i}$ in the $(t-1)$-th game. That proves the lemma we need.

(Iterated weak dominance): Notice that the above lemma is true even if we apply iterated weakly dominance instead of strictly dominance, without changes in the proof. Thus, $\sigma$ is the Nash equilibrium of a game if it is a Nash equilibrium of the game in which strategies have been removed by iterated weak dominance.

Backward Induction

The first pirate should propose $(0, 1, 0, 1)$ to the pirate 2,3,4,5 respectively, and leaves himself 98 coins. This proposal is justified by backward induction. Consider the subgame in which pirate 4 proposes. He will leave 0 to pirate 5 because himself alone can pass the proposal. In the subgame when
pirate 3 proposes, he will leave (0,1) to pirate 4 and 5 respectively. Pirate 5 should accept because otherwise he will end up with getting 0 coins.\(^2\) Given that result, when pirate 2 proposes he should leave (0,1,0) to pirate 3, 4, 5 respectively, for he only need one more vote, and pirate 4 should accept, since otherwise pirate 3 will leave 0 to him. Based on the above reasoning, the first pirate should propose (0,1,0,1) and expects pirate 3 and 5 to accept.

**Correlated Equilibrium**

Algebraically, we need to check the following four inequalities:

\[
\frac{1}{2}u_1(U,l) + \frac{1}{2}u_1(U,r) \geq \frac{1}{2}u_1(D,l) + \frac{1}{2}u_1(D,r),
\]

\[
u_1(D,l) \geq u_1(U,l),
\]

\[
\frac{1}{2}u_2(U,l) + \frac{1}{2}u_2(D,l) \geq \frac{1}{2}u_2(U,r) + \frac{1}{2}u_2(D,r),
\]

\[
u_2(U,r) \geq u_2(U,l).
\]

And the argument is as follows. The correlated equilibrium assigns probability 1/3 to \((U,l), (U,r), \text{ and } (D,l)\). Suppose that player 2 follows the equilibrium strategy. If player 1 receives a signal of playing \(U\), then she expects that player 2 will equally mix between \(l\) and \(r\). Then playing \(U\) gives her 1, which is equal to the payoff of playing \(D\). Thus playing \(U\) is her best reply. If she receives \(D\), then she expects that player 2 will play \(l\), and in such case playing \(D\) is strictly better than playing \(U\), so again following the signal is a best reply. The same argument applies to player 2 since the game is symmetric. That shows the proposed strategy profile is really a correlated equilibrium.

\(^2\) How about pirate 4? Given that pirate 5 will accept the offer, pirate 4’s action doesn’t matter. However it is reasonable for him to reject, for in case pirate 5 is making a mistake by rejecting the offer then he can get 100 coins. I will apply this argument to all other subgames.
1. Elicitation of Beliefs

We assume that the experimental subject’s optimization problem is to maximize expected reward (where expectations are calculated with respect to the true subjective probabilities, \( \mu \)) by choosing a report \( \nu \), which must lie in \( \Delta (\Omega) = \{ \nu \in \mathbb{R}^\Omega : \nu(\omega) \geq 0, \sum \nu(\omega) = 1 \} \). Therefore, given a reward structure \( r : \Omega \times \Delta (\Omega) \to \mathbb{R} \), the subject’s problem is to

\[
(1.1) \quad \max \sum_{\omega \in \Omega} r(\omega, \nu) \mu(\omega) : \nu \in \Delta (\Omega) \}
\]

If \( r(\omega, \nu) = \nu(\omega) \) then the Lagrangian associated with (1.1) is given by (with \( \lambda \) denoting the Lagrange multiplier associated with the constraint \( \sum \nu(\omega) = 1 \))

\[
(1.2) \quad L(\nu, \lambda) = \sum_{\omega \in \Omega} \nu(\omega) \mu(\omega) + \lambda \left[ 1 - \sum_{\omega \in \Omega} \nu(\omega) \right]
\]

The Kuhn-Tucker conditions for this problem are, for each \( \omega \in \Omega \),

\[
(1.3) \quad \frac{\partial L}{\partial \nu(\omega)} = \mu(\omega) - \lambda \leq 0, \quad \nu(\omega) \geq 0, \quad [\mu(\omega) - \lambda] \nu(\omega) = 0.
\]

Let \( \Omega^*_\mu = \arg \max \mu(\omega) = \{ \omega^* \in \Omega : \mu(\omega^*) \geq \mu(\omega), \forall \omega \in \Omega \} \). Since \( \mu \in \Delta (\Omega) \), there is a state \( \hat{\omega} \) such that \( \mu(\hat{\omega}) > 0 \), so by (1.3), it follows that

\[
\lambda = \mu(\omega^*) = \max_{\omega \in \Omega} \mu(\omega) > 0,
\]

where \( \omega^* \in \Omega^*_\mu \). Therefore, for any \( \omega \notin \Omega^*_\mu \), there is a strict inequality in the Kuhn-Tucker conditions for \( \nu(\omega) \), so \( \nu(\omega) = 0 \). Finally, by inspection, any \( \nu \in \Delta (\Omega^*_\mu) \) leads to same value of the objective function in (1.1), so the set of optimum reports is completely characterized by \( \Delta (\Omega^*_\mu) \). In conclusion, the reward structure \( r(\omega, \nu) = \nu(\omega) \) in general does not elicit true subjective beliefs. For instance, if \( \Omega^*_\mu \) is a singleton \( \{ \omega^* \} \) then the subject’s optimum report is \( \nu(\omega) = 1 \) if \( \omega = \omega^* \) and \( \nu(\omega) = 0 \) otherwise.

If, on the other hand, \( r(\omega, \nu) = \ln(\nu(\omega)) \), the subject’s new optimization problem leads to the Lagrangian

\[
(1.4) \quad L(\nu, \lambda) = \sum_{\omega \in \Omega} \ln(\nu(\omega)) \mu(\omega) + \lambda \left[ 1 - \sum_{\omega \in \Omega} \nu(\omega) \right]
\]

with Kuhn-Tucker conditions

\[
(1.5) \quad \frac{\partial L}{\partial \nu(\omega)} = \frac{\mu(\omega)}{\nu(\omega)} - \lambda \leq 0, \quad \nu(\omega) \geq 0, \quad [\frac{\mu(\omega)}{\nu(\omega)} - \lambda] \nu(\omega) = 0.
\]

Since the objective function is strictly concave and the constraints are linear, it follows the Kuhn-Tucker Theorem that there is a unique solution to the
subject’s optimization problem. Hence, it suffices to find a solution to (1.5) to characterize the subject’s optimum response. To that end, assume that the Kuhn-Tucker conditions hold with equality. In that case, we have that for any two states $\omega, \omega' \in \Omega$,

\[(1.6) \quad \frac{\mu(\omega)}{\nu(\omega)} = \frac{\mu(\omega')}{\nu(\omega')} = \lambda,\]

and by the Equal Ratio Rule (together with the condition that $\mu, \nu \in \Delta(\Omega)$),

\[(1.7) \quad \lambda = \frac{\sum \mu(\omega)}{\sum \nu(\omega)} = 1,\]

which implies that $\mu(\omega) = \nu(\omega)$ for every $\omega \in \Omega$: the subject *optimally reports the truth*.

### 2. Risk Aversion in the Lab

Consider an agent who is indifferent between the gamble, $g$, paying $9.75, -3.00, and -2.25 with equal probability and staying with her initial wealth, $x$. The expected payoff from this gamble is 1.50. The variance of the gamble is 34.125. The agent is indifferent between keeping her wealth and taking the gamble, that is,

\[(2.1) \quad u(x) = E[u(x + \overline{g} + \sigma y)],\]

where $\overline{g}$ denotes the expected monetary payoff from the gamble, $1.5, \sigma$ is the standard deviation of the gamble, and $y$ corresponds to a normalised gamble with mean zero, a variance of unity, and such that $g = \overline{g} + \sigma y$. Defining $z = x + \overline{g}$, we may express the indifference of our agent as

\[(2.2) \quad u(z - \overline{g}) = E[u(z + \sigma y)];\]

this allows us to conclude that $p = [-u''(z)/u'(z)]\sigma^2/2$, the agent’s risk premium, is precisely $\overline{g} = 1.5$. (Can you see why?) Assuming that the agent’s utility function is CES, with CRRA coefficient $\rho = -u''(z)/u'(z)$, it follows that $\overline{g} = \rho \sigma^2/2z$, so we can extract $\rho$ from $z$ and vice-versa. Thus, $\rho = 30,769$ when $z = 350,000$ and $z = 227.5$ when $\rho = 20$.

### 3. The AK Model

The planner wants to solve the following dynamic programming problem:

\[V(k) = \max_{k'} c^{1-\rho}/(1-\rho) + \delta V(k') \quad \text{s.t.} \quad k = c + k'/\beta.\]

The first-order and envelope conditions for this problem yield

\[c^{-\rho}/\beta = \delta V'(k') = \delta c^{1-\rho} \quad \Rightarrow \quad c'/c = (\delta \beta)^{1/\rho}.\]
Let $\gamma := (\delta\beta)^{1/\rho}$ denote the optimum steady-state growth rate of consumption, and $\gamma_k$ denote that of capital. In the steady state, $c_t = \gamma'_c \bar{c}$ and $k_t = \gamma'_k \bar{k}$. Substituting this into the resource constraint, we obtain that $\gamma'_k \bar{k} = \gamma'_c \bar{c} + \gamma'_k (1 - \gamma_k / \beta) \bar{k} = \gamma'_c \bar{c}$. For this to hold for all $t$ at the steady state, we require that $\gamma_k = \gamma_c = \gamma$ and that $(1 - \gamma / \beta) \bar{k} = \bar{c}$. The steady state is reached immediately, since consumption grows at a constant rate from the beginning and the storage technology is linear.

To calculate the initial value of the capital stock, we need to work out $V'_0(k)k$ evaluated at $k = 1$; this represents the increase in lifetime utility associated with a marginal increment in capital, or the (shadow) price of capital that would make the current level optimal, multiplied by the quantity of capital. By the Envelope Theorem, $V'_0(k)k \equiv c \rho$. We also have that $c = (1 - \gamma / \beta)k$, which, after substitution, yields $V'_0(1) = (1 - \gamma / \beta)^{-\rho} = (1 - \gamma / \beta)^{-\rho} = (1 - \delta\beta)^{1/\rho} / \beta^{-\rho}$.

The derivative of the value of the capital stock with respect to $\beta$ can be calculated as

$$\partial V'(1) / \partial \beta = (V'(1))^{1+1/\rho} \gamma (\rho - 1) / \beta^2.$$ 

This is negative if $\rho > 1$, assuming that $\gamma = (\delta\beta)^{1/\rho} < \beta$.1

4. INVESTMENT

The state space is $\Omega = \{b, w\}$, where $b$ stands for ‘bankrupt’ and $w$ for ‘wealthy.’ Denote by $V : \Omega \rightarrow \mathbb{R}$ the investor’s value function. Clearly, we have that $V(b) = 0$. Then, the investor’s problem is to solve

$$V(w) = \max \{2 + \delta p V(b) + \delta (1 - p) V(w), 1 + \delta V(w)\} = \max \{2 + \delta (1 - p) V(w), 1 + \delta V(w)\},$$

where the first term corresponds to investing in stocks and the second to bonds. If it is optimal to invest in stocks then

(4.1) $V(w) = 2 + \delta (1 - p) V(w),$ 

therefore $V(w) = 2 / (1 - \delta (1 - p))$. Investing in stocks will be the optimum decision whenever

(4.2) $2 + \delta (1 - p) V(w) \geq 1 + \delta V(w).$

Rearranging this expression, and substituting for $V(w)$ calculated above, yields investing in stocks as the optimum decision whenever $2\delta p / (1 - \delta (1 - p)) \leq 1.$

---

1Does this assumption make sense to you?