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Probability Theory, Conditional Probability and Bayes Law

Probability Space

A probability space $\omega \in \Omega$ “outcomes”

Events are subsets $E \subset \Omega$

A probability measure is a function defined on events $\mu(E)$

- $\mu(E) \geq 0$
- $\mu(\Omega) = 1$
- if $E \cap F = \emptyset$ then $\mu(E \cup F) = \mu(E) + \mu(F)$

Example

$$\Omega = \{\text{Jane, Mary, John}\}$$

$$\mu\{\text{Mary}\} = 1/2, \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$$

What is $\mu(\{\text{Mary, Jane}\})$?

Random Variables

A random variable $x:\Omega \rightarrow \mathfrak{R}$ assigns each point in the probability space a real number

For example the probability space is

$\Omega = \{\text{Jane, Mary, John}\}$

the random variable is "height"

notice that it makes perfectly good sense to add, subtract, multiply, etc.
random variables

probabilities of random variables are computed from the underlying
probability space

for example

$$x(\text{Jane}) = 5, x(\text{Mary}) = 5, x(\text{John}) = 6$$

$$\mu\{\text{Mary}\} = 1/2, \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$$

what is the probability of 5?

What is the probability that $2x$ is 12?

Expectation

The expectation of a random variable is the probability weighted average

$$Ex = \sum_{\omega \in \Omega} x(\omega) \mu(\omega)$$

for example

$$x(\text{Jane}) = 5, x(\text{Mary}) = 5, x(\text{John}) = 6$$

$$\mu\{\text{Mary}\} = 1/2, \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$$

what is E_x ?

Some important facts

$$E(x + y) = Ex + Ey$$

if a is a constant

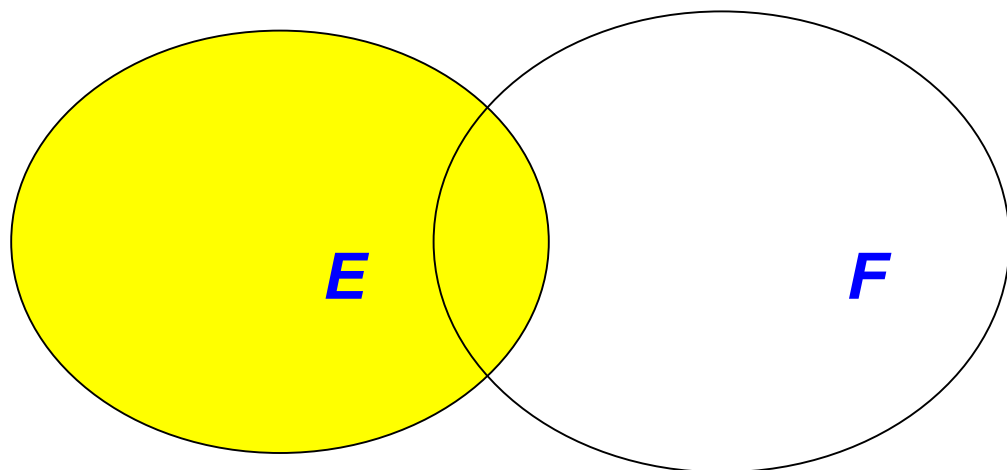
$$Eax = aEx$$

but if y is a random variable

$$Eyx \neq EyEx \text{ in general}$$

Conditional Probability

$$\Pr(E|F) = \Pr(E \cap F) / \Pr(F)$$



Bayes Law

$$\Pr(E|F) = \frac{\Pr(F|E) \Pr(E)}{\Pr(F)}$$

“likelihood” times “prior”

Drug Testing

A drug test has a 5% chance of error. A group of parolees is given the test. Of the parolees, 60% are drug users. If the test is positive how likely is it the parolee is using drugs?

E=using drugs

F=positive test

$$\begin{aligned}\Pr(E|F) &= \frac{\Pr(F|E) \Pr(E)}{\Pr(F)} \\ &= \frac{.95 \times .6}{.95 \times .6 + .05 \times .4} = .97\end{aligned}$$

Now the test is given to a group of airline pilots of whom only 2% are drug users. If the test comes out positive how likely is it the pilot is using drugs?

$$\Pr(E|F) = \frac{.95 \times .02}{.95 \times .02 + .05 \times .98} = .28$$

- cancer testing

The Ann Landers Problem

Ann Landers says that all heroin users once used marijuana, so that if you use marijuana, you will surely end up using heroin

E=heroin use

F=marijuana use

$$\begin{aligned}\Pr(E|F) &= \frac{\Pr(F|E) \Pr(E)}{\Pr(F)} \\ &= \frac{\Pr(E)}{\Pr(F)}\end{aligned}$$

so that if there are 100 times as many marijuana users as heroin users, using marijuana means only a 1% chance of using heroin