Probability Theory, Conditional Probability and Bayes Law

Probability Space

A probability space $\omega \in \Omega$ "outcomes"

Events are subsets $E \subset \Omega$

A probability measure is a function defined on events $\mu(E)$

- $\mu(E) \geq 0$
- $\mu(\Omega) = 1$
- if $E \cap F = \varnothing$ then $\mu(E \cup F) = \mu(E) + \mu(F)$

Example

$\Omega = \{\text{Jane, Mary, John}\}$

$\mu\{\text{Mary}\} = 1/2,\ \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$

What is $\mu(\{\text{Mary,Jane}\})$?

Random Variables

A random variable $x : \Omega \rightarrow \Re$ assigns each point in the probability space a real number

For example the probability space is

$\Omega = \{\text{Jane, Mary, John}\}$

the random variable is "height"

notice that it makes perfectly good sense to add, subtract, multiply, etc. random variables

probabilities of random variables are computed from the underlying probability space *for example*

$x(\text{Jane}) = 5, x(\text{Mary}) = 5, x(\text{John}) = 6$

$\mu\{\text{Mary}\} = 1/2,\ \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$

what is the probability of 5?

What is the probability that 2x is 12?

Expectation

The expectation of a random variable is the probability weighted average

$$Ex = \sum_{\omega \in \Omega} x(\omega)\mu(\omega)$$

for example

$$x(\text{Jane}) = 5, x(\text{Mary}) = 5, x(\text{John}) = 6$$

$$\mu\{\text{Mary}\} = 1/2,\ \mu\{\text{Jane}\} = 1/4, \mu\{\text{John}\} = 1/4$$

what is Ex?

Some important facts

$$E(x+y) = Ex + Ey$$

if *a* is a constant

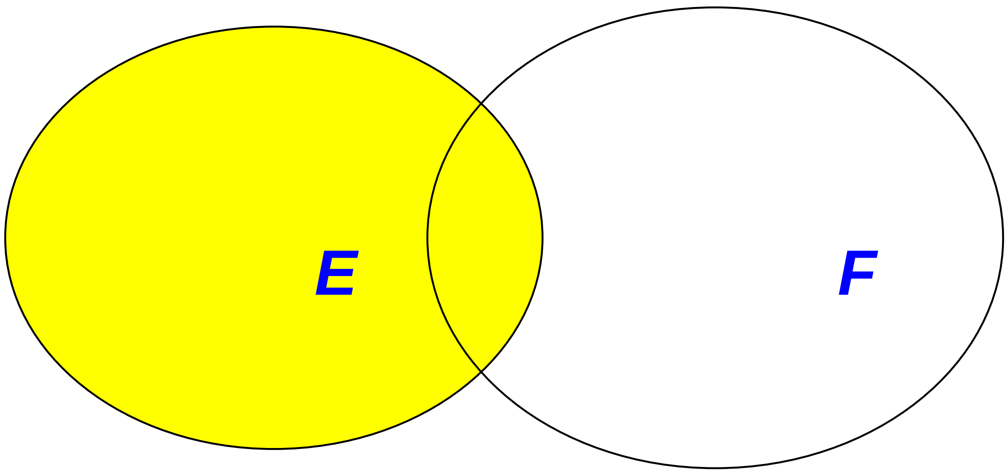
$$Eax = aEx$$

but if *y* is a random variable

$Eyx \neq EyEx$ in general

Conditional Probability

$$\Pr(E|F) = \Pr(E \cap F) / \Pr(F)$$



Bayes Law

$$\Pr(E|F) = \frac{\Pr(F|E)\Pr(E)}{\Pr(F)}$$

“likelihood” times “prior”

Drug Testing

A drug test has a 5% chance of error. A group of parolees is given the test. Of the parolees, 60% are drug users. If the test is positive how likely is it the parolee is using drugs?

E=using drugs

F=positive test

$$\Pr(E|F) = \frac{\Pr(F|E)\Pr(E)}{\Pr(F)}$$

$$= \frac{.95 \times .6}{.95 \times .6 + .05 \times .4} = .97$$

Now the test is given to a group of airline pilots of whom only 2% are drug users. If the test comes out positive how likely is it the pilot is using drugs?

$$\Pr(E|F) = \frac{.95 \times .02}{.95 \times .02 + .05 \times .98} = .28$$

- cancer testing

The Ann Landers Problem

Ann Landers says that all heroin users once used marijuana, so that if you use marijuana, you will surely end up using heroin

E=heroin use

F=marijuana use

$$\Pr(E|F) = \frac{\Pr(F|E)\Pr(E)}{\Pr(F)}$$

$$= \frac{\Pr(E)}{\Pr(F)}$$

so that if there are 100 times as many marijuana users as heroin users, using marijuana means only a 1% chance of using heroin